Mathematics

Senior 4

Student's Book

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present senior four Mathematics book for students of advanced level where Mathematics is a major subject. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;

- Worked examples; and
- Application activities which are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work-related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the editing of this book, particularly, REB staffs and teachers for their technical support.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. Nelson MBARUSHIMANA

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development and the editing of senior four Mathematics book for students of advanced level where Mathematics is a major subject. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to Curriculum Officers and teachers whose efforts during the editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook production.

Joan MURUNGI

Head of Curriculum, Teaching and learning Resources Department

Contents

Topic area: Trigonometry

Sub-topic area: Trigonometric circles and identities

Unit 1

Fundamentals of trigonometry

Key unit competence

Use trigonometric circle and identities to determine trigonometric ratios and apply them to solve related problems.

1.0 Introductory activity

The angle of elevation of the top of the Cathedral from a point 280 m away from the base of its steeple on level ground is 60°. By using trigonometric concepts learnt in senior three, find the height of the cathedral

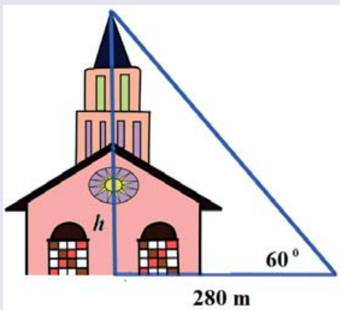


Fig 1.0

1.1 Trigonometric concepts

The word trigonometry is derived from two Greek words: *trigon*, which means triangle, and *metric*, which means measure. So we can define trigonometry as measurement in triangles.

Angle and its measurements

Activity 1.1

In pairs, discuss what an angle is. Sketch different types of angles and name them: acute, obtuse, reflex, etc. Measure the angles to verify the sizes.

An **angle** is the opening that two straight lines form when they meet. In Figure 1.1, when the straight line FA meets the straight line EA, they form the angle we call angle FAE. We may also call it "the angle at the point A," or simply "angle A."

The two straight lines that form an angle are called its **sides**. And the size of the angle does not depend on the lengths of its sides.

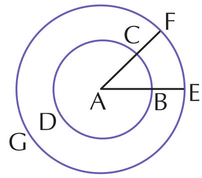


Fig 1.1

Degree measure

To measure an angle in degrees, we imagine the circumference of a circle divided into 360 equal parts. We call each of those equal parts a "degree." Its symbol is a small o: 1° = "1 degree."

From the Figure 1.2, the measure of an angle, will be as many degrees as its sides diverge. To say that angle BAC is 30° means that its sides enclose 30 of those equal divisions. Arc BC is $\frac{30}{180}$ of the entire circumference.

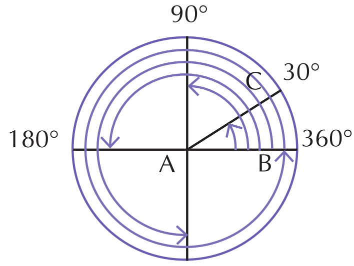


Fig 1.2

Radians

Activity 1.2

This activity will be carried out in the field. Each group will need: two pointed sticks, two ropes (each about 50 cm in length), black board protractor and metre rule.

1. Fix the pointed sticks, one at each end of the rope.
2. Place the sharp end of one of the sticks onto the ground.
3. With that point as the centre, let the tip of the 2^{nd} stick draw a circle, radius 50 cm.
4. Take the 2^{nd} rope also of length 50 cm and fit it on any part of the circumference (an arc).
5. Take the metre rule and draw a line from each of the ends of the rope to the centre of the circle.
6. Use the large protractor to measure the angle enclosed by the two lines. What do you get?
7. Compare your result with the rest of the groups. Are they almost similar?
8. Repeat the task using ropes of length 70 cm each. How do the results compare with the ones of 50 cm lengths?

9.

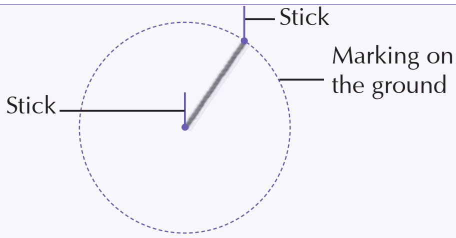


Fig 1.3

The radian is a unit of angular measure. It is defined such that an angle of one radian subtended from the centre of a **unit circle** produces an arc with **arc length** of r.

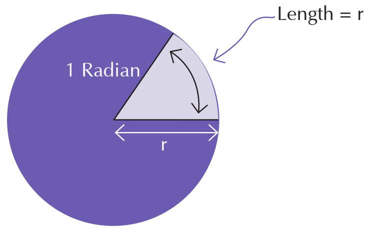


Fig 1.4

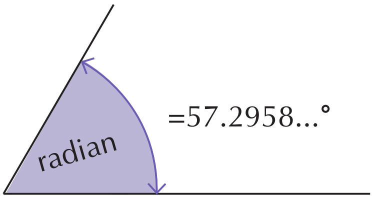


Fig 1.5

1 radian is *about* 57.2958 **degrees**. The **radian** is a pure measure based on the **radius** of the circle.

Degree-radian conversions

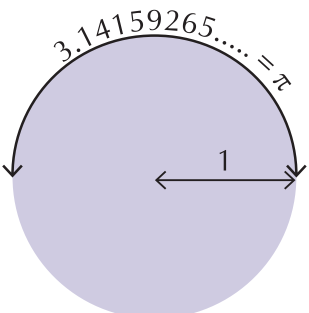


Fig 1.6

There are π radians in a half circle and 180° in the half circle.

So π radians = 180° and 1 radian = $\frac{180°}{\pi}$ = 57.2958° (approximately).

A full circle is therefore 2π radians. So there are 360° per 2π radians, equal to $\frac{180°}{\pi}$ or 57. 29577951°/radian. Similarly, a right angle is $\frac{\pi}{2}$ radians and a straight angle is π radians.

Table 1.1 Degree to radian equivalents

Degrees	Radians (exact)	Radians (approx)
30°	$\frac{\pi}{6}$	0.524
45°	$\frac{\pi}{4}$	0.785
60°	$\frac{\pi}{3}$	1.047
90°	$\frac{\pi}{2}$	1.571
180°	π	3.142
270°	$\frac{3\pi}{2}$	4.712
360°	2π	6.283

The following diagram is useful for converting from one system of measure to the other:

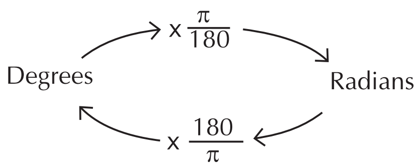


Fig 1.7

Example 1.1

Convert 45° to radians in terms of π.

Solution

$45° = (45 \times \frac{\pi}{180})$ radians $= \frac{\pi}{4}$ radians.

Example 1.2

Convert $\frac{5\pi}{6}$ to degrees.

Solution

$\frac{5\pi}{6} = \frac{5\pi \times 180°}{6\pi} = 150°$.

Application activity 1.1

1. Convert the following degree measures to radians, in terms of π:

a)	90°	b)	60°	c)	30°	d)	18°
e)	9°	f)	135°	g)	225°	h)	270°
i)	360°	j)	720°	k)	315°	l)	540°
m)	36°	n)	80°	o)	230°		

2. Convert the following radian measures to degrees:

a) $\frac{\pi}{5}$ b) $\frac{3\pi}{5}$ c) $\frac{3\pi}{4}$ d) $\frac{\pi}{18}$

e) $\frac{\pi}{9}$ f) $\frac{7\pi}{9}$ g) $\frac{\pi}{10}$ h) $\frac{3\pi}{20}$

i) $\frac{5\pi}{6}$ j) $\frac{\pi}{8}$

3. Copy and complete the tables:

a)

Degrees	0	45	90	135	180	225	270	315	360
Radians									

b)

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

1.1.2 The unit circle

Activity 1.3

Imagine a point on the edge of a wheel. As the wheel turns, how high is the point above the centre?Represent this using a drawing.

The unit circle is the circle with centre (0, 0) and radius 1 unit.

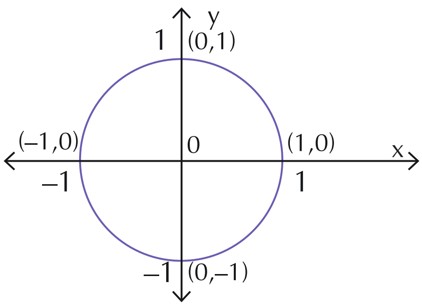


Fig 1.8

1.1.3 Definition of sine and cosine in unit circle

Activity 1.4

Consider a point P(x,y) which lies on the unit circle in the first quadrant. OP makes an angle θ with the x-axis as shown in Figure 1.9.

Draw a figure similar to Figure 1.9 on the Cartesian plane. The radius is 1 unit.

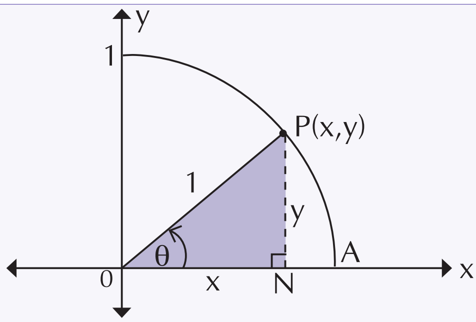


Fig 1.9

Pick any point $P(x,y)$. What is the value of $\frac{x}{1}$ and $\frac{y}{1}$ Pick different points on the circumfrence and calculate $\frac{x}{1}$ and $\frac{y}{1}$. What do you notice?

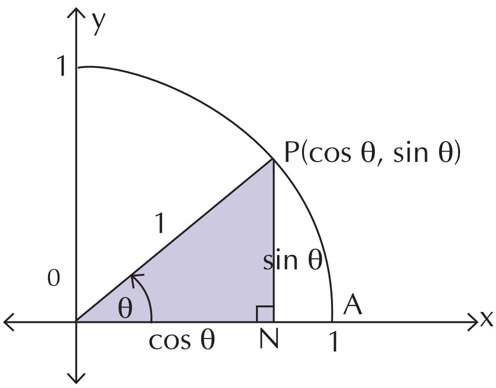
The x is the side adjacent to the angle θ. And y is the side opposite the angle θ.The radius of 1 unit is the hypotenuse.

Using right-angled triangle trigonometry:

$$\cos x = \frac{\text{Adjacent side}}{\text{Hypotenuse side}} = \frac{x}{1} = x$$

$$\sin x = \frac{\text{Opposite side}}{\text{Hypotenuse side}} = \frac{y}{1} = y$$

$$\tan \theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$



On the quarter circle, if ∠AOP= θ°, then the coordinates of P are (cos θ, sin θ).

The x-and y-coordinates of P each have a special name.

- The y-coordinate is called **the sine of angle** θ and noted sin θ.
- The x-coordinate is called **the cosine of angle** θ and noted cos θ.

Notice also that in the triangle ONP, $x^2 + y^2 = 1$ (Pythagoras) and so

$(\cos \theta)^2 + (\sin \theta)^2 = 1$ or $\cos^2 \theta + \sin^2 \theta = 1$

Note: We use $\cos^2 \theta$ for $(\cos \theta)^2$ and $\sin^2 \theta$ for $(\sin \theta)^2$.

Activity 1.5

Use graph paper to draw a circle of radius 10 cm. Measure the half chord and the distance from the centre of the chord. Use various angles, say for multiples of 15°. Plot the graphs and determine the sines and the cosines.

We can also use the quarter unit circle to get another ratio. This is tangent which is written as 'tan' in short.

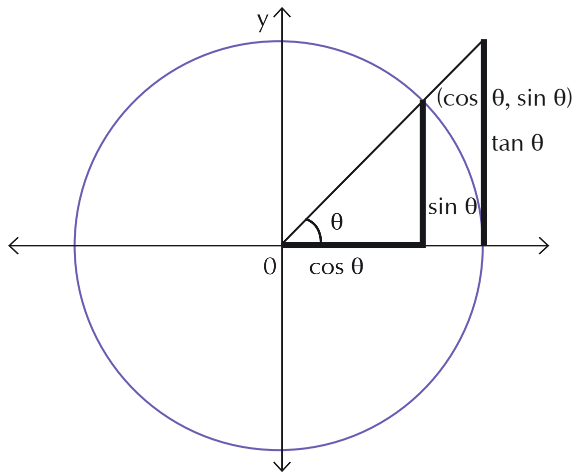


Fig 1.11

1.1.4 Trigonometric ratios

Activity 1.6

Carry out research on the trigonometric ratios. What are they?
Define them.

There are three basic trigonometric ratios: **sine**, **cosine**, and **tangent**. The other common trigonometric ratios are: **secant, cosecant** and **cotangent.**

Trigonometric ratios in a right-angled triangle

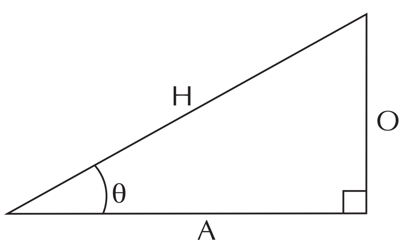


Fig 1.12

In Figure 1.12, the side H opposite the right angle is called the **hypotenuse**. Relative to the angle θ, the side O opposite the angle θ is called the **opposite side**. The remaining side A is called the **adjacent side**.

Trigonometric ratios provide relationships between the sides and angles of a right angle triangle. The three most commonly used ratios are sine, cosine and tangent.

Sine	$\sin\theta = \frac{O}{H}$
Cosine	$\cos\theta = \frac{A}{H}$
Tangent	$\tan\theta = \frac{O}{A}$

Note that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Other ratios are defined by using the above three:

Cosecant	$\csc\theta = \frac{1}{\sin\theta} = \frac{H}{0}$
Secant	$\sec\theta = \frac{1}{\cos\theta} = \frac{H}{A}$
Cotangent	$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{A}{0}$

These six ratios of the lengths of the sides of a right triangle are called **trigonometric functions** of acute angles. They are independent of the unit used.

The trigonometric ratios of the angles 30 , 45 and 60 are often used in mechanics and other branches of mathematics. So it is useful to calculate them and know their values by heart.

The angle 45

Activity 1.7

Draw an isosceles right -angled triangle where the two equal sides are 1 unit in length. Use Pythagoras' theorem to calculate the hypotenuse. Deduce the three primary trigonometric ratios of 45^0 (cosine, sine and tangent).

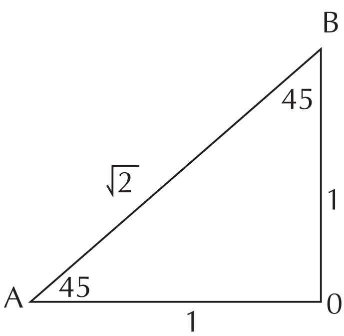


Fig 1.13

In Figure 1.13, the triangle is isosceles. Hence the opposite side and adjacent sides are equal, say 1 unit.

The hypotenuse is therefore of length $\sqrt{2}$ units (using Pythagoras' theorem).

We have

$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan 45° = 1$

The angles 60 and 30

Activity 1.8

In pairs, draw an equilateral triangle, ABC, of sides 2 units in length. Then draw a line AD from A perpendicular to BC. AD bisects BC giving BD = CD = 1. Refer to the previous activity and do the following: From ΔBDA, determine the trigonometric ratios of 60°

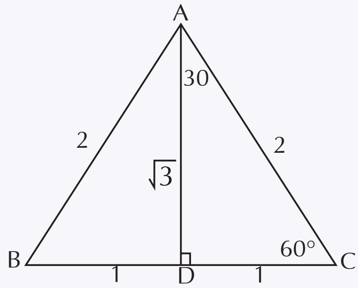


Fig 1.14

From this we can determine the following trigonometric ratios for the special angles 30 and 60 :

$\sin 60° = \frac{\sqrt{3}}{2}$	$\sin 30° = \frac{1}{2}$
$\cos 60° = \frac{1}{2}$	$\cos 30° = \frac{\sqrt{3}}{2}$
$\tan 60° = \sqrt{3}$	$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

We can now complete the following table.

θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–

1.2 Reduction formulae on positive acute angles

1.2.1 Complementary angles

Two angles are complementary if their sum is 90° $(= \frac{\pi}{2})$.

These two angles (30° and 60°) of Figure 1.15 are **complementary angles**, because they add up to 90°. Notice that together they make a **right angle** ∟. Sin (30°) $= \frac{1}{2}$ = cos (60°);

sin (60°) $= \frac{\sqrt{3}}{2}$ = cos (30°).

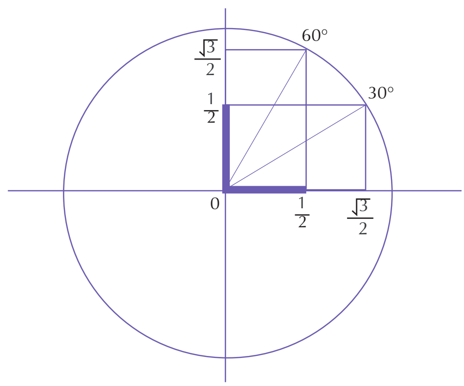


Fig 1.15

Thus, given an acute angle θ $(\frac{\pi}{2} - \theta)$ and θ are complementary angles.

For the two complementary angles $(\frac{\pi}{2} - \theta)$ and θ, we have the following:

$\sin(\frac{\pi}{2} - \theta) = \cos\theta$ $\quad$ $\tan(\frac{\pi}{2} - \theta) = \cot\theta$

$\cos(\frac{\pi}{2} - \theta) = \sin\theta$ $\quad$ $\cot(\frac{\pi}{2} - \theta) = \tan\theta.$

1.2.2 Supplementary angles

Two angles are supplementary if their sum is 180° (= π).

The two angles (45° and 135°) of Figure 1.16 are supplementary angles because they add up to 180°.

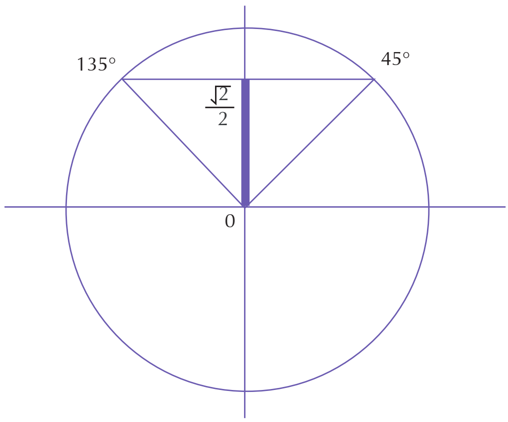


Fig 1.16

Notice that together they make a **straight angle**. Sin (135°) = $\frac{\sqrt{2}}{2}$ = sin (45°).

Thus, $(\pi - \theta)$ and θ are supplementary angles.

For the two supplementary angles $(\pi - \theta)$ and θ, we have the following:

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\cot(\pi - \theta) = -\cot\theta$.

Opposite angles

Two angles are opposite if their sum is 0.

Thus, $-\theta$ and θ are opposite angles.

For the two opposite angles $-\theta$ and θ, we have the following:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\cot(-\theta) = -\cot\theta$.

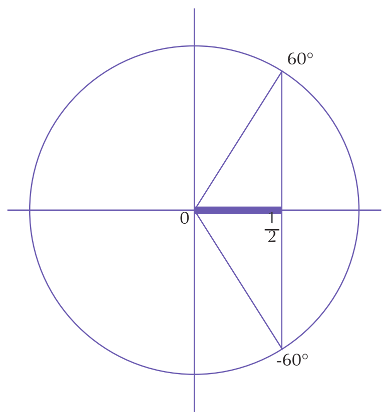


Fig 1.17

1.2.4 Anti-complementary angles

Two angles are anti-complementary if their difference is 90° (= $\frac{\pi}{2}$).

Notice that the two angles (60° and 150°) of Figure 1.18 are anti-complementary angles because their difference is 90°. Sin (150°) = $\frac{1}{2}$ = cos (60°).

Thus, given the acute angle θ ($\frac{\pi}{2}$ + θ) and θ are anti-complementary angles.

For the two anti-complementary angles ($\frac{\pi}{2}$ + θ) and θ, we have the following:

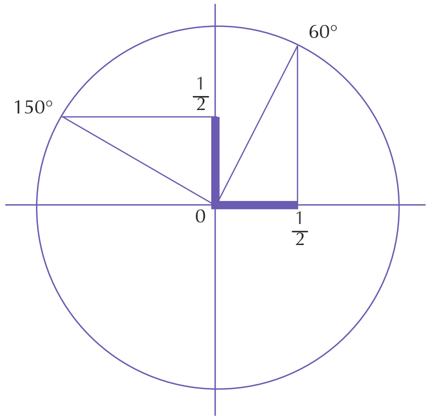


Fig 1.18

$\sin(\frac{\pi}{2} + \theta) = \cos\theta$ $\tan(\frac{\pi}{2} + \theta) = -\cot\theta$

$\cos(\frac{\pi}{2} + \theta) = -\sin\theta$ $\cot(\frac{\pi}{2} + \theta) = -\tan\theta$.

1.2.5 Anti-supplementary angles

Two angles are anti-supplementary if their difference is 180° (= π).

Notice that the two angles (30° and 210°) of Figure 1.19 are anti-supplementary angles, because their difference is 180°.

Sin (210°) = $-\frac{1}{2}$ = – sin (30°).

Thus, given the acute angle θ (π + θ) and θ are anti-supplementary angles.

For the two anti-supplementary angles (π + θ) and θ, we have the following:

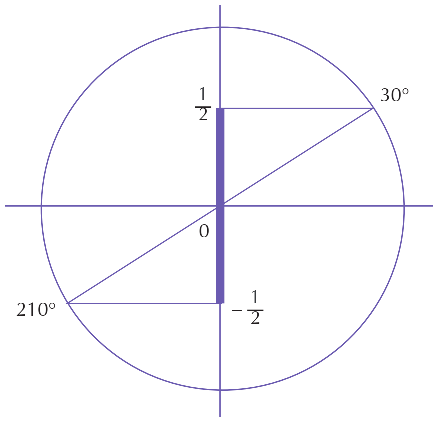


Fig 1.19

$\sin(\pi + \theta) = -\sin\theta$ $\tan(\pi + \theta) = \tan\theta$

$\cos(\pi + \theta) = -\cos\theta$ $\cot(\pi + \theta) = \cot\theta$.

Example 1.3

Evaluate the following:

a) $\sin\frac{5\pi}{6}$ b) $\tan\frac{5\pi}{4}$ c) $\cos\frac{2\pi}{3}$ d) tan (–30°)

Solution

a) $\sin\frac{5\pi}{6} = \sin(\pi - \frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$

b) $\tan\frac{5\pi}{4} = \tan(\pi + \frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

c) $\cos\frac{2\pi}{3} = \cos(\pi - \frac{\pi}{3}) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

d) $\tan(-30°) = -\tan 30° = -\frac{\sqrt{3}}{3}$

1.2.6 Coterminal angles

Coterminal angles are angles in standard position (angles with the initial side on the positive *x*-axis) that have a common terminal side. For example 30°, –330° and 390° are all coterminal.

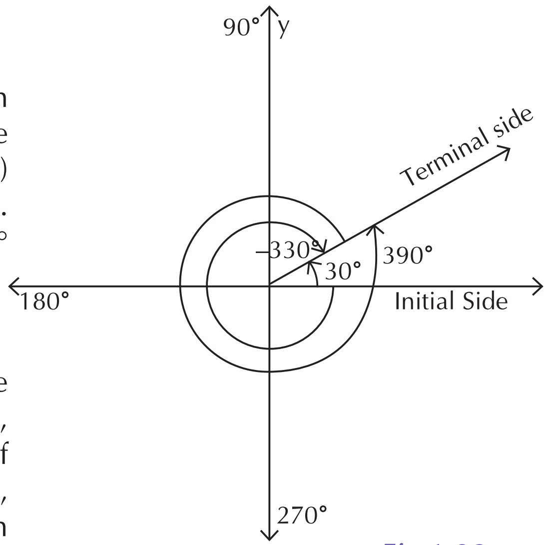


Fig 1.20

To find a positive and a negative coterminal angle with a given angle, you can add and subtract 360°, if the angle is measured in degrees, or 2π if the angle is measured in radians.

Example 1.4

Find a positive and a negative angle coterminal with a 55° angle.

Solution

$55° - 360° = -305°$

$55° + 360° = 415°$

A –305° angle and a 415° angle are coterminal with a 55° angle.

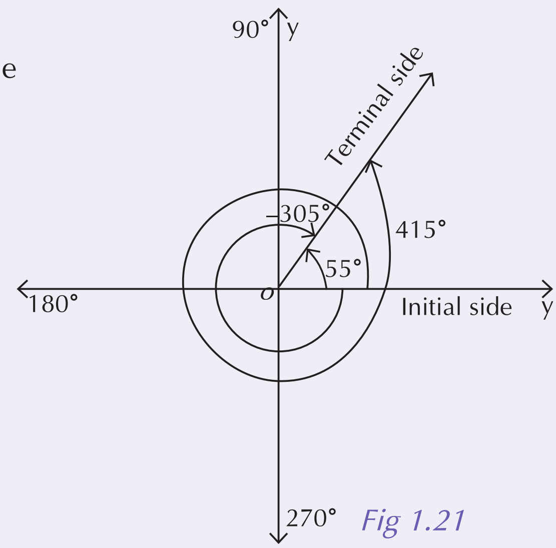


Fig 1.21

Example 1.5

Find a positive and a negative angle coterminal with a $\frac{\pi}{3}$ angle.

Solution

$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$

$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$

A $\frac{7\pi}{3}$ angle and a $-\frac{5\pi}{3}$ angle are coterminal with a $\frac{\pi}{3}$ angle.

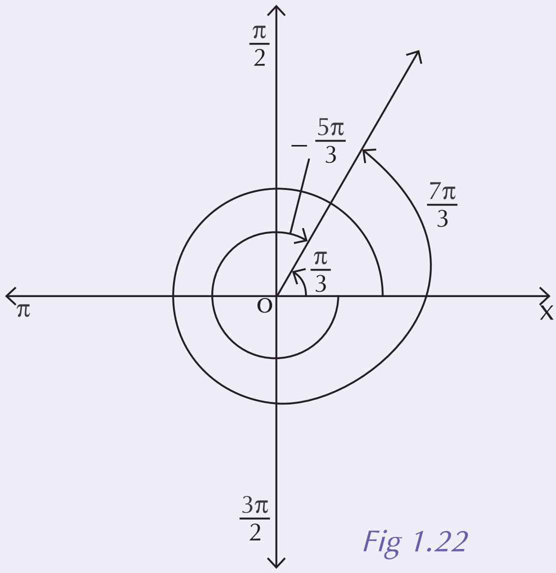


Fig 1.22

Application activity 1.2

1. Determine, if possible, the trigonometric ratios of

   (a) 0° (b) 90° (c) 45° (d) 120°

2. (a) Given $\sin\theta = 0.8$, find the possible values of $\cos\theta$ and $\tan\theta$.

   (b) Given $\cos\theta = 0.5$, find the possible values of $\sin\theta$ and $\tan\theta$.

   (c) Given $\tan\theta = -2$, find the possible values of $\sin\theta$ and $\cos\theta$.

3. Simplify the following expressions:

   (a) $\cos\theta\tan\theta$ (b) $\frac{\sin\theta}{\tan\theta}$

   (c) $\frac{\cos\theta\tan\theta + \sin\theta}{\tan\theta}$ (d) $\tan\theta\left(\cos\theta + \frac{\sin\theta}{\tan\theta}\right)$

   (e) $\frac{1}{\tan\theta}(\sin\theta + \cos\theta\tan\theta)$

4. Find in each of the following, four possible values of θ for which

   (a) $\sin\theta = 0$ (b) $\cos\theta = 0$ or $\sin\theta = 0$

   (c) $\tan\theta$ does not exist.

1.3 The trigonometric identities

In mathematics, an **identity** is an equation which is always true. There are many trigonometric identities, but the one you are most likely to see and use is,

$\sin^2\theta + \cos^2\theta = 1$

Proof

P(cos θ, sin θ) lies on the unit circle $x^2 + y^2 = 1$.

Therefore $(\cos\theta)^2 + (\sin\theta)^2 = 1$ or $\sin^2\theta + \cos^2\theta = 1$

Other formulas of trigonometric ratios

Application activity 1.3

The following are other trigonometric identities. Use the unit circle to prove that the identities are correct.

1. $1 + \tan^2\theta = \sec^2\theta$ $(\cos\theta \neq 0)$
2. $1 + \cot^2\theta = \csc^2\theta$ $(\sin\theta \neq 0)$
3. $\tan\theta + \cot\theta = \sec\theta\csc\theta$
4. $\sin^2 A - 4\cos^2 A + 1 = 3\sin^2 A - 2\cos^2 A - 1$
5. $\dfrac{\cos^2 A}{1 + \tan^2 A} - \dfrac{\sin^2 A}{1 + \cot^2 A} = 1 - 2\sin^2 A$

Application activity 1.4

Simplify the following:

(a) $\sin^2 2A + \cos^2 2A$

(b) $1 + \tan^2 \frac{A}{4}$

(c) $\sin^2 B + \cos^2 B$

(d) $\cos^2\theta + 1$

(e) $\cos^2 4A + \sin^2 4A$

(f) $\cos^2 1\frac{1}{2} + \sin^2 1\frac{1}{2}$

(g) $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta}$

(h) $(\sin A + \cos A)^2 + (\sin A - \cos A)^2$

(i) $1 - \sin^2 A$

(j) $1 - \cos^2 2B$

(k) $\sec^2\theta - 1$

(l) $1 - \csc^2 A$

1.4 Graphs of some trigonometric functions

1.4.1 Sine function: sin x = y

Activity 1.9

You are invited to draw a graph of the function $y = \sin x$. Start by preparing a table of values by choosing suitable values of x . Then, find the corresponding values of y .

2. For the graph of y = sin x in $-\pi \le x \le \pi$, consider the following table. Note that the function is periodic with period 2π.

x	…	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	…
y = sin x	…	0	-0.71	-1	-0.71	0	0.71	1	0.71	0	…

3. Plot the ordered pairs (x, sin x). Join these points with a smooth curve to obtain the following graph:

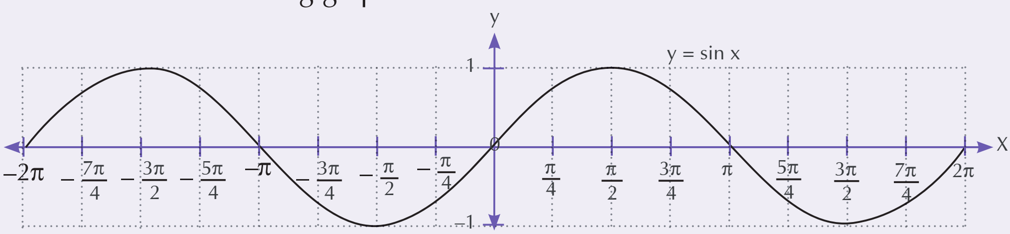


Fig 1.23

1.4.2 Cosine function: cos x = y

Activity 1.10

To draw a graph of the function y = cos x . Start by completing a table of values for x and y.

For the graph of y = cos x in $-\pi \le x \le \pi$, consider the following table. Note that the function is periodic with period 2π.

x	…	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	…
y = cos x	…	–1	-0.71	0	0.71	1	0.71	0	-0.71	-1	…

Plot the ordered pairs (x, cos x) and then join these points with a smooth curve. The following graph is obtained:

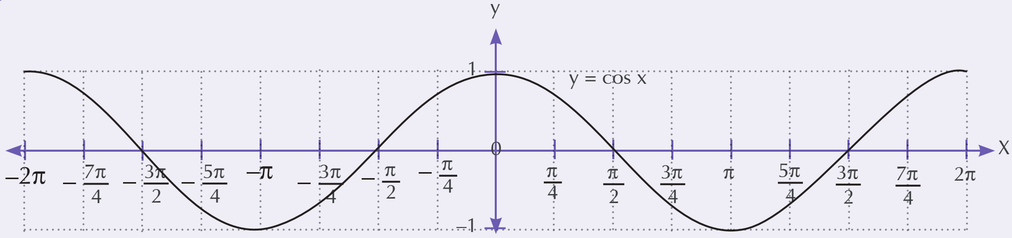


Fig 1.24

1.4.3 Tangent function: tan x = y

Activity 1.11

Draw a graph of the function y = tan x, prepare a table by choosing suitable values of x. Then find the corresponding values of y.

For the graph of y = tan x in $-\pi \le x \le \pi$, note that the function is periodic with period π.

In addition,

Since $\tan x = \frac{\sin x}{\cos x}$, tan x, will be undefined whenever cos x = 0. The zeros of the function y = cos x correspond to vertical asymptotes of the function y = tan x.

x	…	−π	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	…
y = tan x	…	0	1	$-\infty$	−1	0	1	$+\infty$	−1	0	…

Plot the ordered pairs (x, tan x) and then join these points with a smooth curve. The following graph is obtained:

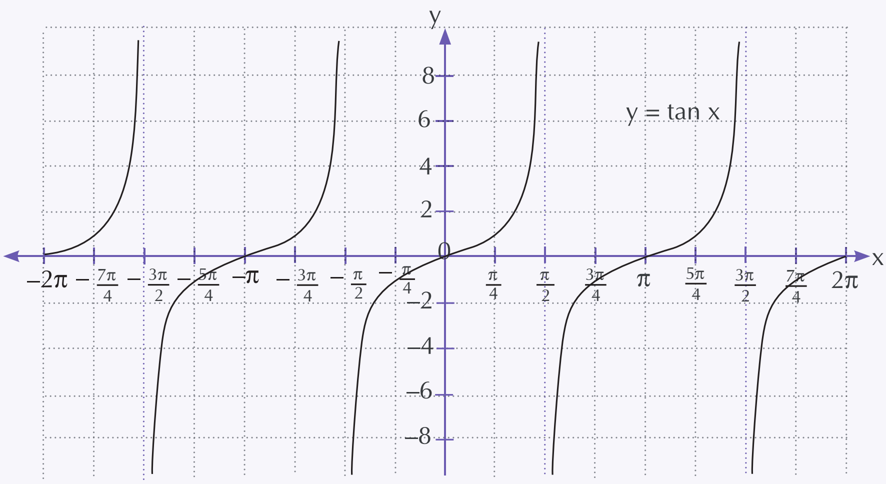


Fig 1.25

1.5 Triangles and applications

1.5.1 Triangles

The cosine rule

For any triangle with sides a, b, c and angles ABC as shown in Figure 1.26, the cosine rule is applicable:

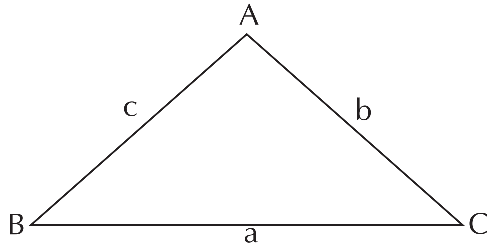


Fig 1.26

$a^2 = b^2 + c^2 - 2bc \cos A$ or

$b^2 = a^2 + c^2 - 2ac \cos B$ or

$c^2 = a^2 + b^2 - 2ab \cos C$

Note: For a right triangle at A i.e. A = 90°, cos A = 0. So $a^2 = b^2 + c^2 - 2bc \cos A$; reduces to $a^2 = b^2 + c^2$, and is the Pythagoras'Rule.

The cosine rule can be used to solve triangles if we are given:

- Two sides and an included angle
- Three sides.

Example 1.12

Find the length of BC.

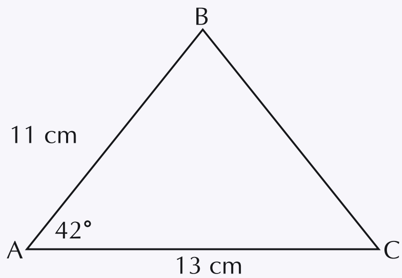


Fig 1.27

Solution

By cosine rule:

$(BC)^2 = 11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42°$

$BC = \sqrt{11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42°}$

BC = 8.801cm

Rearrangement of the original cosine rule formulae can be used to find angles if we know all three sides. The formulae for finding angles are:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 1.13

Find the three angles of the triangle shown in Figure 1.28.

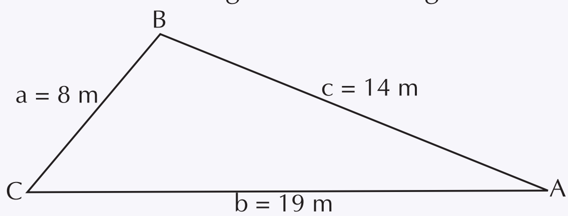


Fig 1.28

Solution

It is a good idea to first find the angle opposite the longest side, side b in this case. Using the alternative form of the law of cosines:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{8^2 + 14^2 - 19^2}{2(8)(14)} \approx -0.45089$$

B = 116.8°

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{19^2 + 14^2 - 8^2}{2(19)(14)} = \frac{361 + 196 - 64}{532} = \frac{493}{532} = 0.9267$$

A = 22.07°

$$\cos A = \frac{a^2 + b^2 - c^2}{2ab} = \frac{8^2 + 19^2 - 14^2}{2(8)(19)} = \frac{64 + 361 - 196}{304} = \frac{229}{304} = 0.7533$$

C = 41.12°

Example 1.14

Find the measure of angle A in Figure 1.29.

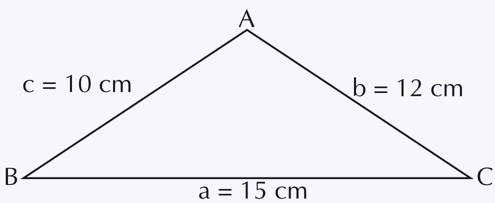


Fig 1.29

Solution

By the cosine rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(12^2 + 10^2 - 15^2)}{(2x\ 12\ x\ 10)} = \frac{144 + 100 - 225}{240} = \frac{19}{240} \approx 0.079$$

$A = \cos^{-1}(0.079) = 85.87°$

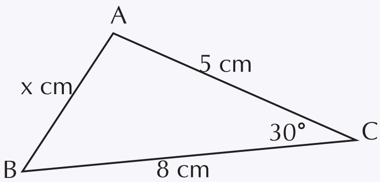
Thus, angle A measures 85.87°

Note: To solve the triangle is to find all its measures of sides and angles.

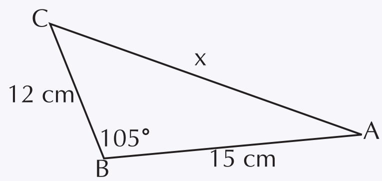
Application activity 1.8

1. Find the lengths of the unknown side x in the given triangles:

a)



b)



c)

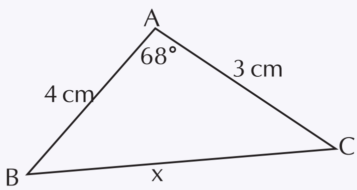
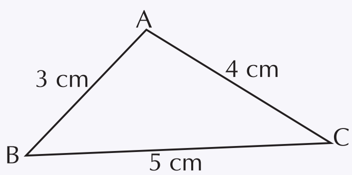


Fig 1.30

2. Find the sizes of all angles in the triangles below:

a)



b)

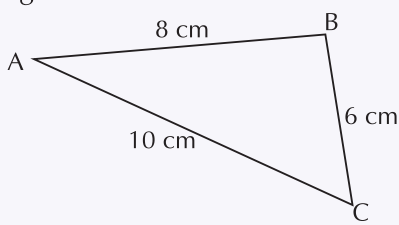


Fig 1.31

3. Solve the triangles with the following sides:

   a) a = 10 cm, b = 8 cm, c = 12 cm   b) a = 6 cm, b = 5 cm, c = 7 cm

4. Solve the triangles with the following measures:

   a) a = 5 cm, b = 6 cm, C = 45°   b) a = 12 cm, B = 57°, c = 15 cm

   c) B = 117°, a = 3.4 cm, c = 2.7 cm   d) B = 60° , a = 12 cm, c = 15 cm

The sine rule

In any triangle ABC with sides a, b and c units in length, and opposite angles A, B and C respectively,

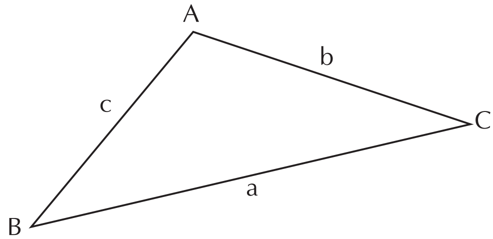


Fig 1.32

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note: The sine rule is used to resolve problems involving triangles given either:

- Two angles and one side, or
- Two sides and a non-included angle.

Example 1.15

Find the length of AC in Figure 1.33.

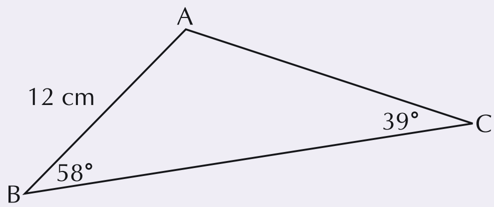


Fig 1.33

Solution

Using the sine rule, $\frac{b}{\sin 58°} = \frac{12}{\sin 39°}$

$$b = \frac{12 \times \sin 58°}{\sin 39°}$$

$b = 16.17$

AC = 16.17 cm

Example 1.16

Find the measure of angle C in triangle ABC if AC is 7 cm, AB is 11 cm and angle B measures 25°.

Solution

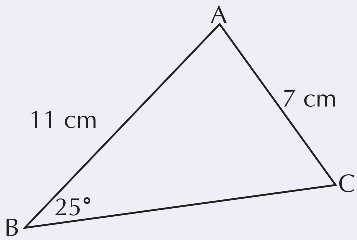


Fig 1.34

By the sine rule $\frac{\sin C}{c} = \frac{\sin B}{b}$

$\frac{\sin C}{11} = \frac{\sin 25°}{7}$

$\sin C = \frac{11 \times \sin 25°}{7}$

$C = \sin^{-1}\left(\frac{11 \times \sin 25°}{7}\right)$ – the supplement of angle C

C = 41.6° or 180° – 41.6°.

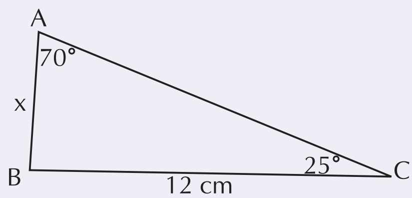
However, the angle C can also be an obtuse angle.

Thus, C = 41.6° or C = 138.4°

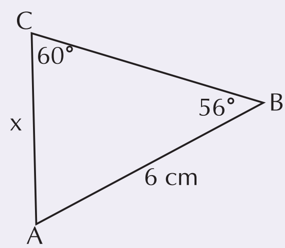
Application activity1.9

1. Find the measures of the unknown sides x in the triangles below.

a)



b)



c)

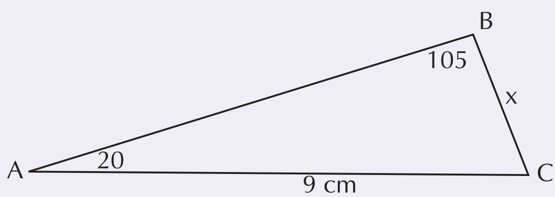


Fig 1.35

2. Find the measures of all angles and sides in the two triangles below.

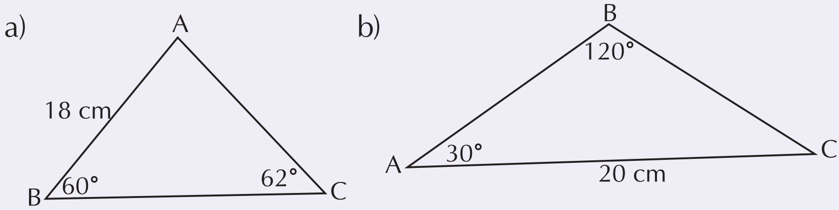


Fig 1.36

3. Solve the triangles with the following measurements:
   a) A = 52°, a = 6 cm, B = 67°
   b) B = 38°, b = 5 m, C = 48°
   c) C = 71°, c = 19 cm, b = 5 cm
4. A triangle of vertices ABC has the following measures: B = 40°, b = 8 cm and c = 11 cm. Find the two possible values for the angle C.
5. For the triangle ABC, find the measure of the angle:
   a) A if a = 14.6 cm, b = 17.4 cm and B = 65°
   b) C if a = 6.5 m, c = 4.8 m and A = 71°
   c) B if b=43.8 cm, c = 31.4 cm and C = 43°.

1.5.2 Air navigation

Example 1.17

An airplane flying horizontally 1 km above the ground is observed at an elevation of 60°. If after 10 seconds the elevation is observed to be 30°, find the uniform speed per hour of the airplane.

Solution

Let E be the first position of the aeroplane and D be the position after 10 seconds.

EB = 1 km and DC = 1 km. Angle of elevation EAB=60°

Angle EAB = 60°. Angle DAC = 30°

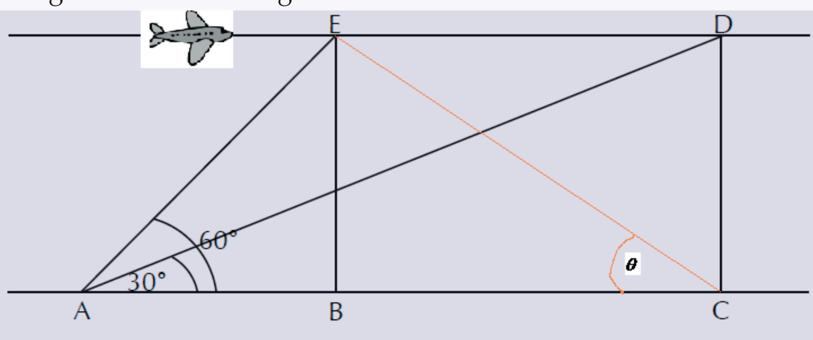


Fig 1.37

$\text{Tan } 60° = \frac{BE}{AB} \Rightarrow AB = \frac{BE}{\tan 60°} = \frac{1}{\sqrt{3}}$

$\text{Tan } 30° = \frac{CD}{AC} \Rightarrow AC = \frac{CD}{\tan 30°} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}$

Distance travelled by aeroplane is ED = BC = AC – AB = $\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

Uniform speed of the aeroplane is:

$$\text{Velocity} = \frac{\text{distance}}{\text{time}} = \frac{\frac{2}{\sqrt{3}}}{10} = \frac{2}{\sqrt{3}} \times \frac{1}{10} = \frac{2}{5\sqrt{3}} = \frac{\sqrt{3}}{15} \text{ km/s}$$

$$= \frac{\sqrt{3}}{15} \times 3600 \text{ km/h} = 240\sqrt{3} \text{ km/h} = 415.69 \text{ km/h}$$

The angle ECB formed from the airplane to the foot of its direction D is called angle of depression

1.5.3 Inclined plane

Example 1.18

A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 45^0. When he retreats 50 metres from the bank, he finds the angle to be 30^0. Find the breadth of the river and the height of the tree.

Solution

Let AB = h metres be the height of the tree and CB = x metres be the breadth of the river

$\angle BCA = 45^o$.

Consider C as the first position of the person and D as the second position of the person after he retreats.

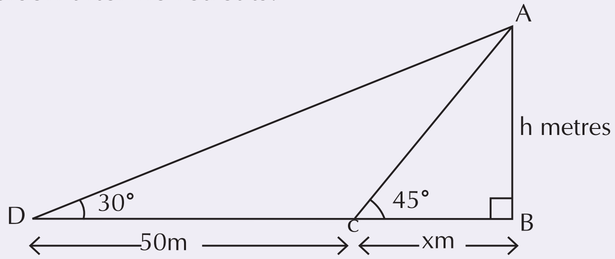


Fig 1.38

Therefore, $\angle BDA = 30^o$

$\angle ABC = \angle ABD = 90^o$

$\text{Sin } 45° = \frac{AB}{AC}$ and $\cos 45° = \frac{BC}{AC}$

$\therefore \text{Tan } 45° = \frac{\sin 45°}{\cos 45°} = \frac{AB}{BC}$ where AB = h and BC = x.

$\frac{h}{x} = \tan 45°$

$\Leftrightarrow \frac{h}{x} = 1$

$h = x$ .... (1)

$\frac{AB}{BD} = \tan 30°$

$\Leftrightarrow \frac{h}{30 + x} = \tan 30°$

$\Leftrightarrow \frac{h}{30 + x} = \frac{1}{\sqrt{3}}$

$\sqrt{3}h = 30 + x$ .... (2)

Putting (1) into (2), we get

$\sqrt{3}(x) = 30 + x$

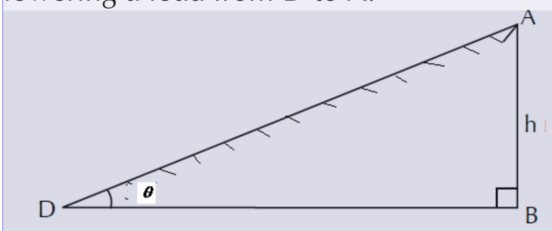
$\Leftrightarrow \sqrt{3}x = 30 + x$

$\Leftrightarrow (\sqrt{3} - 1)x = 30$

$x = \frac{30}{\sqrt{3} - 1} \approx 40.98$ m

Then h ≈ 40.98 m.

An inclined plane, also known as a ramp, is a flat supporting surface tilted at an angle , with one end A higher than the other D , used as an aid for raising or lowering a load from D to A.



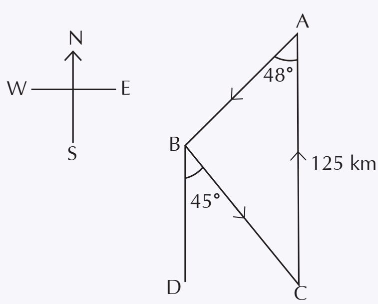
$\sin\theta = \frac{AB}{AD}$

1.5.4 Bearing

Example 1.19

The course for a boat race starts at point A and proceeds in the direction S 48° W to point B. Then in the direction S 45° E to point C, and finally back to A, as illustrated in Figure 1.39.

Point C lies 125 km directly south of point A. Approximate the total distance of the race course.



Solution

Because lines BD and AC are parallel, it follows that ∠ BCA = ∠ BCA = 45°

Consequently, triangle ABC has the measures shown in Figure 1.40.

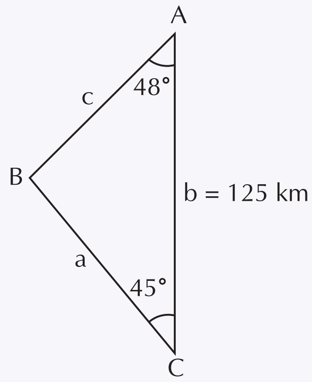


Fig 1.40

For angle B, we have B =180°– (48°+45°) = 87°.

Using the Law of Sine:

$$\frac{\sin 48°}{a} = \frac{\sin 87°}{b} = \frac{\sin 45°}{c}$$

$$\frac{\sin 48°}{a} = \frac{\sin 87°}{125}$$

$$125 \sin 48° = a \sin 87°$$

$$a = \frac{125 \sin 48°}{\sin 87°} = \frac{92.89}{0.99} = 93.83$$

$$\frac{\sin 87°}{b} = \frac{\sin 45°}{c}$$

$$c = \frac{b \sin 45°}{\sin 87°} = \frac{125 \sin 45°}{\sin 87°} = \frac{88.39}{0.99} = 89.28$$

The total length of the course is approximately

125 km + 93.83 km + 89.28 km = 308.11 km

Application activity 1.10

1. A tree is located on an incline of a hill. The tree is broken and the tip of the tree touches the hill farther down the hill and forms an angle of 30° with the hill. The broken part of the tree and the original tree form an angle of 50° at the break. The original part of the tree is 3 m tall. How tall was the tree before it broke?
2. Two ships are located 200 m and 300 m respectively from a lighthouse. If the angle formed by their paths to the lighthouse is 96°. What is the distance between the two ships?
3. Two cars leave the same station at the same time, moving along straight tracks that form an angle of 30°. If one car travels at an average speed of 50 km/hour and the other at an average speed of 60 km/hour, how far apart are the two cars after two hours?

4. From a tower of 32 m of height, a car is observed at an angle of depression of 55°. Find how far the car is from the tower.
5. A town B is 13 km south and 18 km west of a town A. Find the bearing and distance of B from A.
6. A tower of 30 m of height stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the tower is 33°. From the same point the angle of elevation to the bottom of the tower is 32°. Find the height of the hill.
7. An aeroplane is 1200 m directly above one end of a field. The angle of depression of the other end of the field from the aeroplane is 64°. How long is the field?
8. Mutesi is standing on the bank of a river and observes that the angle subtended by a tree on the opposite bank is 60°. When she retreats 40 m from the bank, she finds the angle to be 30°. Find the height of the tree and the breadth of the river.
9. From a ship the angle of elevation of a point A at the top of a cliff is 12°. After the ship has sailed 400 m directly towards the foot of the cliff, the angle of elevation of A is 45°. Find the height of the cliff.
10. From a tower 60 m high, the angles of depression of two billboards which are in a horizontal line through the base of the tower are 20° and 25° respectively. Find the distance between the billboards if they are on:
    (a) the same side of the tower
    (b) opposite sides of the tower.
11. An aeroplane takes off at a constant angle of 20°.

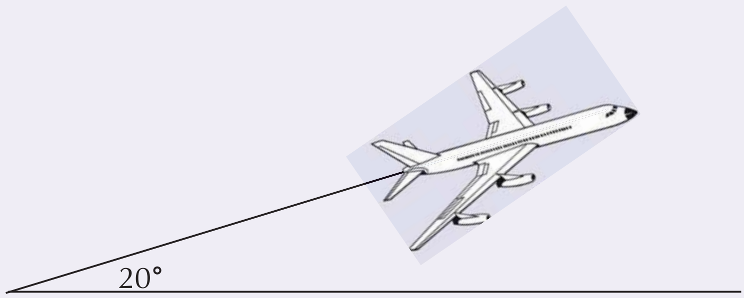


Fig 1.41

By the time it has flown 1000 m, what is its altitude? Give your answer correct to the nearest metre.

12. An aeroplane flying at an altitude of 10 000 m is directly overhead. Two minutes later it is at an angle of 38° to the horizontal. How fast is the aeroplane flying, in kilometres per hour?

Summary

1. The **radian** is a unit of angular measure. It is defined such that an angle of one radian subtended from the centre of a **circle** of radius r produces an arc with **a length** of r units.
2. The **unit circle** is the circle with centre (0, 0) and radius 1 unit.
3. There are three basic trigonometric ratios: **sine**, **cosine**, and **tangent**. The other common trigonometric ratios are: **secant, cosecant and cotangent.**
4. Two angles θ and $(\frac{\pi}{2} - \theta)$ are **complementary** if their sum is $\frac{\pi}{2}$, in this case $\sin(\frac{\pi}{2} - \theta) = \cos\theta$
5. Two angles θ and $(\pi - \theta)$ are **supplementary** if their sum is π. in this case $\sin(\pi - \theta) = \sin\theta$
6. Two angles θ and $-\theta$ are **opposite** if their sum is 0 degree, in this case $\cos\theta = \cos(-\theta)$
7. Two angles $(\frac{\pi}{2} + \theta)$ and θ are **anti-complementary** if their difference is $\frac{\pi}{2}$, in this case $\sin(\frac{\pi}{2} + \theta) = \cos\theta$.
8. Two angles θ and $(\pi + \theta)$ are **anti-supplementary** if their difference is π, $\sin(\pi + \theta) = -\sin\theta$
9. **Coterminal angles** are angles in standard position (angles with the initial side on the positive *x*-axis) that have a common **terminal** side.

Topic area: Algebra

Sub-topic area: Mathematical logic and applications

Unit 2

Propositional and predicate logic

Key unit competence

Use mathematical logic to organise scientific knowledge and as a tool of reasoning in daily life.

2.0 Introductory activity

1. Discuss whether the following logical argument is valid or not valid. Justify your answer

"If you give a child an orange and another child an orange. The two children get an orange".

2. From the following expression, give your answer by true or false

a. Every integer larger than 1 is positive.

b. Kampala is in Rwanda.

c. How old are you?

d. Every liquid is water.

e. Write down the names of Rwandan president.

f. $1-x^2=0$.

g. Rwanda is African country or Rwanda is a member of Commonwealth.

2.1 Introduction

Definitions and notation

Activity 2.1

Carry out research to distinguish between a statement and a proposition. Give examples of statements and propositions.

Present your findings in class for discussion.

A **statement** is a declarative sentence or descriptive sentence. A logical **proposition** is a statement that has truth value. It can either be true or false, but not both.

In logic, we seek to express statements, and the connections between them in algebraic symbols. Logic propositions can be denoted by letters such as p, q, r, s, ...

Below are some examples:

- r: $4 < 8$
- s: If $x = 4$ then $x + 3 = 7$
- t: Nyanza is city in Rwanda
- p: What a beautiful evening!

The statements r, s and t are logic propositions. The statement p is not a logic proposition.

Recall that a proposition is a declarative sentence that is either true or false. Here are some more examples of propositions:

- All cows are brown.
- The Earth is farther from the sun than Venus.
- $2 + 2 = 5$.

Here are some sentences that are not propositions.

1. "Do you want to go to the market?"

   Since a question is not a declarative sentence, it fails to be a proposition.
2. "Clean up your room."

   Likewise, this is not a declarative sentence; hence, fails to be a proposition.
3. "$2x = 2 + x$."

   This is a declarative sentence, but unless x is assigned a value or is otherwise prescribed, the sentence is neither true nor false, hence, not a proposition.

Each proposition can be assigned one of two truth values. We use T or 1 for true and use F or 0 for false.

Application activity 2.1

Discuss and state which of the following sentences are propositions. Give reasons for your answers.

(a) $27 + 35 = 62$

(b) January 1 occurs in the winter in the northern hemisphere.

(c) The population of Rwanda is less than 2 million.

(d) Is Jupiter round?

(e) 6 is greater than 16.

(f) *m* is greater than *n*.

2.2 Propositional logic

The simplest and most abstract logic we can study is called propositional logic.

2.2.1 Truth tables values and truth table

The **true** values and **false** values are generally called true values of a logic proposition. At each proposition, corresponds an application that is a set {*True, False*} denoted by {T,F} or {1,0}.

You can use truth tables to determine the truth or falsity of a complicated statement based on the truth or falsity of its simple components.

A statement in sentential logic is built from simple statements using the logical connectives ~, ∧, ∨, ⇒ and ⇔. We can construct tables which show how the truth or falsity of a statement built with these connectives depends on the truth or falsity of its components.

Figure 2.1 is the table for negation:

This table (Figure 2.1) is easy to understand. If p is *true*, its negation ~ *p* is *false*. If p is *false*, then ~ p is true}.

P
T
F

Fig 2.1

Therefore are 2^n different possibilities of truth values given n logic propositions:

i. In the case of one proposition p there are $2^1 = 2$ possibilities
   p : T or F

ii. In the case of two propositions p and q there are $2^2 = 4$ possibilities

p	**q**
T	T
T	F
F	T
F	F

Fig 2.5

iii. The case of three propositions p, q and r there are $2^3 = 8$ possibilities

p	**q**	**r**
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Fig 2.6

2.2.2 Negation of a logic proposition (statement)

Activity 2.2

You are given a statement P:I go to school. What is the negative statement?

Logical negation is represented by the operator "NOT" denoted by "~ or ¬ ". The negation of the proposition p is the proposition ~p which is true if p is false and false if p is true.

For example:

1. p : The lion is a wild animal; ~p. The lion is not a wild animal.
2. q :3 < 7; ~q: 3 ≥ 7

2.2.3 Logical connectives and their truth tables

Activity 2.2

You are given two statements:
p:I study Maths
q:I study English.
What do you say if you combine the two statements?

Propositions are combined by means of connectives such as **and, or, if … then** and **if and only if** and they are modified by **not**.

1. Conjunction

The logical conjunction is represented by the binary operator "**and**" denoted by "∧" . If p and q are propositions, $p \wedge q$ is true only when both p and q are true. See Figure 2.7.

p	**q**	**p ∧ q**
T	T	T
T	F	F
F	T	F
F	F	F

Fig 2.7

2. Disjunction

The logical disjunction is represented by the binary operator "or" denoted by "$\vee$". If p and q are propositions, $p \vee q$ is true when either p or q is true. See Figure 2.8

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Fig 2.8

3. Implication

The statement "p implies q" or "if p then q", written as "$p \Rightarrow q$" is called the implication or the conditional. In this setting, p is called "the premise, hypothesis or antecedent of the implication" and q is "the conclusion or the consequence of the implication". $p \Rightarrow q$ is false only when the antecedent p is true and the consequence q is false

p	q	$P \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

.

Fig 2.9

The statement $p \Rightarrow q$, when p is false, is sometimes called "a vacuous statement". The converse of $p \Rightarrow q$ is the implication $q \Rightarrow p$. If an implication is true, then its converse may or may not be true.

4. Equivalence

Statements p and q are said to be equivalent if they have the same truth table values. The corresponding logical symbols are "$\Leftrightarrow$" and "$\equiv$", and sometimes "iff". The biconditional $p \Leftrightarrow q$ read as "**p is .... if and only if q**" is true only when p and q have the same truth values.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Fig 2.9

Example 2.1

Construct the truth table of: $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$

Solution

p	$\sim p$	q	$p \Rightarrow q$	$\sim p \vee q$	$(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$
T	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
F	T	F	T	T	T

Fig 2.10

Note: The order of operations for the five logical connectives is as follows:

1. $\sim$
2. $\wedge$, $\vee$ in any order
3. $\Rightarrow$, $\Leftrightarrow$, in any order

Application activity 2.2

1. Construct truth tables for each of the following statements:

a) $p \wedge (\sim p)$

b) $\sim [p \wedge (\sim p)]$

c) $p \wedge (\sim q)$

d) $\sim p \vee q$

e) $\sim [p \wedge (\sim q)]$

f) $p \wedge (q \vee r)$

g) $(p \wedge q) \wedge (\sim p)$

h) $\sim [(\sim p) \vee (\sim q)]$

i) $(\sim p \vee q) \wedge ((\sim p \wedge (\sim q))$

2. Verify with the help of truth tables

a) $\sim (p \wedge q) \equiv \sim p \vee (\sim q)$

b) $p \vee q \equiv \sim (\sim p \wedge \sim q)$

3. Construct truth tables for the following statements:

(a) Tuyishimire plays football and Tuyishimire plays netball.

(b) If I work hard, I will pass the examination.

(c) A number is even if and only if it is divisible by 2.

4. (a) Construct the truth table for the negative of the statement "Kalisa plays football."

   (b) Let p be "Nsengimana speaks Kinyarwanda" and q be "Nsengimana speaks French". Give a simple verbal sentence which describes each of the following:

   i. $p \wedge (\sim q)$

   ii. $\sim(\sim p)$

   iii. $\sim p \vee (\sim q)$

   iv. $\sim p \wedge (\sim q)$

   (c) i. Construct the truth table for $\sim(\sim p \wedge q)$

   ii. Write in symbolic form the statement, "Today is Monday and Nyanza Football Club is not playing".

5. Express the following in symbolic form and then draw its truth table.

   "If you go to the market, you will need money or you won't be able to buy anything".

2.2.4 Tautologies and contradictions

A compound statement is a **tautology** if its truth value is always T, regardless of the truth values of its variables. It is a **contradiction** if its truth value is always F, regardless of the truth values of its variables. Notice that these are properties of a single statement, while logical equivalence always relates two statements.

Example 2.2

Show that the following are tautologies:

(a) $p \vee (\sim p)$.

(b) $(p \vee q) \vee [(\sim p) \wedge (\sim q)]$

Solution

(a) We look at its truth table to check this:

p	~p	p ∨ (~p)
T	F	T
F	T	T

↑

All T

Fig 2.11

Since there are only T's in the p ∨ (~p) column, we conclude that p ∨ (~p) is a tautology. We can think of this as saying that the truth value of the statement p∨(~p) is independent of the value of the "input" variable p.

(b) The given statement has the following truth table.

p	q	~p	~q	p ∨ q	~(p) ∧ (~q)	(p ∨ q) ∧ [(~p) ∧ (~q)]
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T
						↑
						All T

Fig 2.12

When a statement is a tautology, we also say that the statement is **tautological**. In common usage this sometimes simply means that the statement is convincing. We are using it for something stronger: the statement is always true, under all circumstances. In contrast, a contradiction, or **contradictory statement**, is never true, under any circumstances.

Example 2.3

Show that the statement (p ∨ q) ∧ (~p ∧ ~q) is a contradiction.

Solution

Its truth table is the following:

p	q	~p	~q	p ∨ q	~(p) ∧ (~q)	(p ∨ q) ∧ [(~p) ∧ (~q)]
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Fig 2.13

Since there are all Fs in the last column, we conclude that (p ∨ q) ∧ (~p ∧ ~q) is a contradiction.

2.2.5 Contingency

Contingency is the status of propositions that are neither true under every possible valuation nor false under every possible valuation. A contingent proposition is neither necessarily true nor necessarily false.

Considering the truth table, a compound statement is a contingent if there is **T** beneath its main connective in at least one row of its truth table, and an F beneath its main connective in at least one row of its truth table.

Example 2.4

Paul is either a Philosopher or a wise man, and he's not a philosopher.

If p: Paul is a philosopher, w: Paul is a wise man, we have:

p	$\sim p$	w	$p \vee w$	$(p \vee w) \wedge \sim p$
T	F	T	T	F
T	F	F	T	F
F	T	T	T	T
F	T	F	F	F

The expressions of the 4th and the 5th columns are contingents.

Application activity 2.3

Construct a truth table for $(p \Rightarrow q) \wedge (q \Rightarrow r)$.[1]

2.2.6 Logically equivalent proposition

Two statements X and Y are **logically equivalent** if $X \Leftrightarrow Y$ is a tautology. Another way to say this is: For each assignment of truth values to the **simple statements** which make up X and Y, the statements X and Y have identical truth values.

From a practical point of view, you can replace a statement in a proof by any logically equivalent statement.

To test whether X and Y are logically equivalent, you could set up a truth table to test whether $X \Leftrightarrow Y$ is a tautology; that is, whether $X \Leftrightarrow Y$ "has all Ts in its column". However, it's easier to set up a table containing X and Y and then check whether the columns for X and for Y are the same.

Example 2.5

Show that $p \Rightarrow q$ and $\sim p \vee q$ are logically equivalent.

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Fig 2.15

Since the columns for $p \Rightarrow q$ and $\sim p \vee q$ are identical, the two statements are logically equivalent. This tautology is called **conditional disjunction**. You can use this equivalence to replace a **conditional** by a **disjunction**.

2.2.7 De Morgan's Law

For any two propositions p and q, the following hold:

1. $\sim(p \vee q) = \sim p \wedge \sim q$
2. $\sim(p \wedge q) = \sim p \vee \sim q$

   Given two sets A, B in the universal set U:

   $(A \cup B)' = A' \cap B'$

   $(A \cap B)' = A' \cup B'$

There are an infinite number of tautologies and logical equivalences. A few examples are given below.

Double negation	$\sim(\sim p) \Leftrightarrow p$
DeMorgan's Law	$\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$
DeMorgan's Law	$\sim(p \wedge q) \Leftrightarrow (\sim p \wedge \sim q)$
Contrapositive	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
Modus ponens	$[p \wedge (p \Rightarrow q)] \Rightarrow q$
Modus tollens	$q \wedge (p \Rightarrow q) \Rightarrow \sim p$

Example 2.6

Write down the negation of the following statements, simplifying so that only simple statements are negated.

Solution

(a) $(p \vee \sim q)$

$\sim(p \vee \sim q) \Leftrightarrow \sim p \wedge \sim(\sim q)$ .... De Morgan's law

$\Leftrightarrow \sim p \wedge q$ .... Double negation

(b) $(p \wedge q) \Rightarrow R$

$\sim[(p \wedge q) \Rightarrow R] \Leftrightarrow \sim [\sim(p \wedge q) \vee r]$ .... Conditional Disjunction

$\Leftrightarrow \sim[\sim (p \wedge q)] \wedge \sim r$ .... De Morgan's law

$\Leftrightarrow (p \wedge q) \wedge \sim r$

We have seen that $(A \rightarrow B)$ and $(\sim A \vee B)$ are logically equivalent, in an earlier example.

Application activity 2.4

Use De Morgan's Law to write the negation of the following statements, simplifying so that only simple statements are negated:

1. Murerwa is not home or Iyakaremye is doing communal work.
2. If Nyirarukundo buys a banana, then Muragijimana buys an orange.

2.3 Predicate logic

2.3.1 Propositional functions

The propositional functions are propositions that contain variables i.e. a sentence expressed in a way that would assume the value of true or false, except that within the sentence there is a variable (x) that is not defined or specified, which leaves the statement undetermined.

The statements such as $x+2 > 5$ are declarative statements but not propositions when the variables are not specified. However, one can produce propositions from such statements. A propositional function or predicate is an expression involving one or more variables defined on some domain, called the domain of discourse. Substitution of a particular value for the variable(s) produces a proposition which is either true or false. For instance, $P(x) : x+2 > 5$ is a predicate on the set of real numbers. Observe that P(2) is false, P(4) is true. In the expression P(x); x is called a free variable. As x varies, the truth value of P(x) varies as well. The set of true values of a predicate P(x) is called the truth set.

Example 2.7

Given the propositional function, $P(x): x + 1 > 5$, determine the values and the truth value for the following:

a) P(8)

b) P(2)

c) $P(8) \wedge [\sim P(2)]$

Solution

a) $P(8): 8 + 1 > 5 \Leftrightarrow P(8): 9 > 5$ (True)

b) $P(2): 2 + 1 > 5 \Leftrightarrow P(2): 3 > 5$ (False)

c) $P(8) \wedge [\sim P(2)]: (8 + 1 > 5) \wedge [\sim (2 + 1 > 5)]$

$P(8) \wedge [\sim P(2)]: (9 > 5) \wedge [\sim (3 > 5)]$ (True)

Activity 2.4

Carry out research to find the meaning of a propositional function. Discuss your findings, using examples, with the rest of the class.

2.3.2 Logic quantifiers

a) **Universal quantifier**

The universal quantifier is the symbol "$\forall$" read as "for all" or "given any". It means that each element of a set verifies a given property defined on that set.

For example, consider the following equation in $\mathbb{R}$: $1x = x$

Every x that belongs to $\mathbb{R}$, verify the equation. In form of equation we write

$\forall x \in \mathbb{R}: 1x = x$.

b) Existence quantifier

The symbol "∃" read as "there exists" is called existence quantifier. It means that we can find at least an element of a set which verifies a given property defined on the set.

For example, consider the following inequality in $\mathbb{R}$: $x + 3 < 5$. In this case, x represents a range of elements which verify the inequality. In form of equation we write

$\exists x \in \mathbb{R}, x + 3 < 5$

The symbol "∃!" read as "there exists only one" is used in the case of unique existence.

Negation of logical quantifiers

Activity 2.5

Consider the statement:

p: All students in this class are brown

Discuss what is required to show the statement is false by explaining and thus negate the statement.

The negation of 'All students in this class are boys' is 'There is a student in this class who is not a boy'.

To negate a statement with a universal quantifier, we change a universal quantifier to the existential one and then negate the propositional function, and vice versa.

The negation of $\forall x : p$ is $\exists x : (\sim p)$ and the negation of $\exists x : p$ is $\forall x : (\sim p)$ and thus:

$\sim (\forall x : p) \equiv \exists x : (\sim p))$

$\sim (\exists x : p) \equiv \forall x : (\sim p)$

Example 2.8

Negate the following statements:

a) p: All prime natural numbers are integers.

b) q:$\forall x > 0 : x^2 > 5$

Solution

a) ~p: There exist a natural number which is not an integer.

b) $\sim q = \sim(\forall x > 0 : x^2 > 5) = \exists x > 0 : x^2 \leq 5$

Thus, the negation of $\forall x > 0 : x^2 > 5$ is $\exists x > 0 : x^2 \leq 5$

Example 2.9

Negate the following statement:

$\forall x, y \in \mathbb{N} : (x \div y) \in \mathbb{N}$

Solution

$\sim p : \exists x, y \in \mathbb{N} : (x \div y) \notin \mathbb{N}$

Note: The negation of $\forall x : p \Rightarrow q$ is $\exists x : p \wedge (\sim q)$

2.4 Applications

Activity 2.6

In pairs, carry out research on the different ways we can apply propositional and predicate logic. Discuss your findings with the rest of the class.

2.4.1 Set theory

Activity 2.7

Carry out research to answer the following:

What is set theory?

How is it an application in logic?

Set theory is the branch of **mathematical logic** that studies **sets**, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics.

In logic, Venn diagrams can be used to represent the truth of certain statements and also for examining logical equivalence of two or more than two statements.

Example 2.10

Give the Venn diagram for the truth of the following statement:

Equilateral triangles are isosceles triangles.

Solution

Let U = The set of all triangles

E = The set of equilateral triangles, and

S = The set of isosceles triangles.

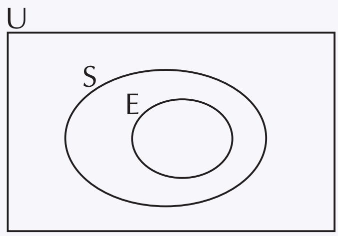


Fig 2.16

Hence, $E \subset U$, $S \subset U$.

Also it is evident from the above statement $E \subset S$.

Task 2.5

1. Represent the truth of each of the following statements by means of a Venn diagram.
   (a) All human beings are mortal and x is a human being.
   (b) No policeman is a thief.
   (c) Every real number is a complex number.
   (d) All natural numbers are rational numbers.
2. Use the Venn diagrams to examine the validity of each of the arguments.
   (a) S_1: All professors are absent-minded
   S_2: Mutesi is not a professor
   S: Mutesi is absent minded.
   (b) S_1: All equilateral triangles are isosceles.
   S_2: T is an equilateral triangle
   S: T is not an isosceles triangle
   (c) S_1: Natural numbers are integers
   S_2: x is an integers
   S: x is not a natural number
   (d) S_1: All professors are absent minded
   S_2: Uwase is not absent minded
   S: Uwase is not a professor

3. Represent the truth of the following statements by means of a Venn diagram:

(a) "Some quadratic equations have two real roots."

(b) "All equilateral triangles are equiangular and all equiangular triangles are equilateral.

4. Use Venn diagrams to examine the validity of each of the arguments.

(a) S_1: All squares are rectangles

S_2: x is not a rectangle

S: x is not a square.

(b) S_1: Natural numbers are integers

S_2: x is an integer

S: x is a natural number.

2.4.2 Electric circuits

If in an electric network two switches are used, then the two switches S_1 and S_2 can be in one of the following two cases;

In series: then the current will flow through the circuit only when the two switches S_1 and S_2 are on (i.e. closed).

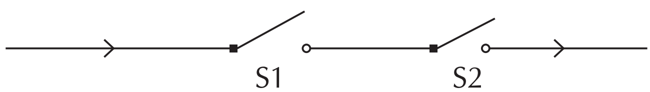


Fig 2.17

In parallel: the current will flow through the circuit if and only if either S_1 or S_2 or both are on (i.e. closed)

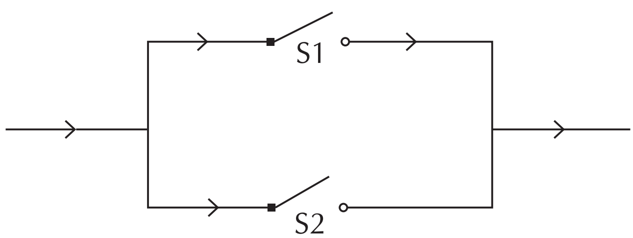


Fig 2.18

Note: If two switches are represented by letters S_1 and S_1', then it means that whenever S_1 is open, S_1' is closed and whenever S_1 is closed, S_1' is open.

Equivalent circuits

Two circuits involving switches S_1, S_2, ..., are said to be equivalent if for every positions of the switches, either current passes through both the circuits, or it does not pass through either circuit.

Example 2.11

Construct a circuit for each of the following:

(a) $p \wedge q$ (b) $p \vee q$

Solution

(a) Let switch S_1 be associated with p and S_2 with q. Then $p \wedge q$ represents a circuit in which S_1 and S_2 are connected in series. The desired circuit is given by the diagram below

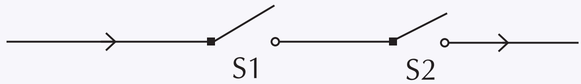


Fig 2.19

The current would flow when both switches S_1 and S_2 are closed.

(b) Here $p \vee q$ represents a circuits in which S_1 and S_2 are connected in parallel. The desired circuit is represented by the diagram below.

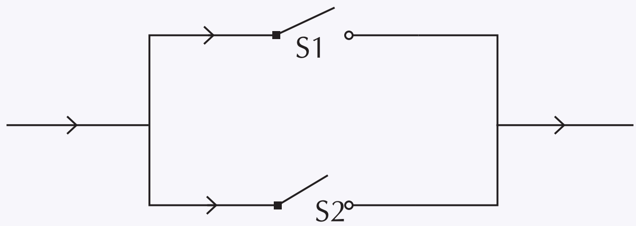


Fig 2.20

The current would flow if and only if at least one of the switches is closed.

Application activity 2.6

Construct a circuit for each of the following statements:

1. $p \vee (q \vee r)$
2. $(p \vee q) \vee r$
3. $p \wedge (q \vee r)$
4. $(p \wedge q) \vee (p \wedge r)$

Table 2.1: Table of set theory symbols

Symbol	Symbol name	Meaning / definition	Example
{ }	set	a collection of elements	A = {3,7,9,14}, B = {9,14,28}
|	such that	so that	$A = \{x \mid x \in \mathbb{R}, x<0\}$
$A \cap B$	intersection	objects that belong to set A and set B	$A \cap B = \{9,14\}$
$A \cup B$	union	objects that belong to set A or set B	$A \cup B = \{3,7,9,14,28\}$

$A \subseteq B$	subset	subset has fewer elements or equal to the set	{9,14,28} $\subseteq$ {9,14,28}
$A \subset B$	proper subset / strict subset	subset has fewer elements than the set	{9,14} $\subset$ {9,14,28}
$A \not\subset B$	not subset	left set not a subset of right set	{9,66} $\not\subset$ {9,14,28}
$A \supseteq B$	superset	set A has more elements or is equal to set B	{9,14,28} $\supseteq$ {9,14,28}
$A \supset B$	proper superset / strict superset	set A has more elements than set B	{9,14,28} $\supset$ {9,14}
$A \not\supset B$	not superset	set A is not a superset of set B	{9,14,28} $\not\supset$ {9,66}
$2^{n(A)}$	power set	all subsets of A	A= {1,2} $2^{n(A)} = 2^2 = 4$ {1,2}, {1}, {2}, { }
A=B	equality	both sets have the same members	A={3,9,14}, B={3,9,14}, A=B
A^c	complement	all the objects that do not belong to set A	A={a,e,i} U= {a,e,i,o,u} A^c = {o,u}
A\B	relative complement	objects that belong to A and not to B	A = {3,9,14}, B = {1,2,3}, A \ B = {9,14}
A - B	relative complement	objects that belong to A and not to B	A = {3,9,14}, B = {1,2,3}, A – B = {9,14}
$A \Delta B$	symmetric difference	objects that belong to A or B but not to their intersection	A = {3,9,14}, B = {1,2,3}, $A \Delta B$ = {1,2,9,14}
$A \ominus B$	symmetric difference	objects that belong to A or B but not to their intersection	A = {3,9,14}, B = {1,2,3}, $A \ominus B$ = {1,2,9,14}
$a \ni A$	element of	set membership	A={3,9,14}, $3 \in A$
$x \notin A$	not element of	no set membership	A={3,9,14}, $1 \notin A$
(a, b)	ordered pair	collection of 2 elements	A={1,2,3} (1,2), (1,3) (2,1)

A x B	Cartesian product	set of all ordered pairs from A and B	A={1,2} B={5} AxB={(1,5), (2,5)}
ǀAǀ or n(A)	cardinality	the number of elements of set A	A={3,9,14}, ǀAǀ=3
#A	cardinality	the number of elements of set A	A={3,9,14}, #A=3
Ш	empty set	Ш = {}	A = Ш
$\mathbb{U}$	universal set	set of all possible values	numerous examples
$\mathbb{N}_0$	natural numbers/ set of whole numbers with zero	$\mathbb{N}_0 = \{0,1,2,3,4,...\}$	$0 \in \mathbb{N}_0$
$\mathbb{N}_1$	natural numbers / whole numbers set (without zero)	$\mathbb{N}_1 = \{1,2,3,4,5,...\}$	$6 \in \mathbb{N}_1$
$\mathbb{Z}$	integer numbers set	$\mathbb{Z} = \{...-3,-2,-1,0,1,2,3,...\}$	$-6 \in \mathbb{Z}$
$\mathbb{Q}$	rational numbers set	$\mathbb{Q} = \{x \mid x=a/b,\ a,b\in\mathbb{Z}\}$	$2/6 \in \mathbb{Q}$
$\mathbb{R}$	real numbers set	$\mathbb{R} = \{x \mid -\infty < x < \infty\}$	$6.343434 \in \mathbb{R}$
$\mathbb{C}$	complex numbers set	$\mathbb{C} = \{z \mid z=a+bi,\ -\infty<a<\infty,\ -\infty<b<\infty\}$	$6+2i \in \mathbb{C}$

Summary

1. A **statement** is a declarative sentence or descriptive sentence.
2. A **logical proposition** is a statement that has truth value. It can either be true or false, but not both.
3. The **true values** and **false values** are generally called true values of a logic proposition. Truth tables determine the truth or falsity of a complicated statement based on the truth or falsity of its simple components.
4. The **logical conjunction** is represented by the binary operator "**and**" denoted by "$\wedge$". If p and q are propositions, $p \wedge q$ is true only when both p and q are true.
5. **The logical disjunction** is represented by the binary operator "or" denoted by "$\vee$". If p and q are propositions, $p \vee q$ is true when either p or q is true.
6. The statement "p implies q" or "If p then q", written as " $p \Rightarrow q$" is called the **implication** or the **conditional**.

7. Statements p and q are said to be **equivalent** if they have the same truth table values.
8. A **tautology** is a propositional function whose truth values are all true. The proposition is valid if it is a tautology.
9. A **contradiction** is a propositional function whose truth value is always false.
10. Contingency is the status of **propositions** that are neither true nor false under every possible valuation.
11. Set theory is the branch of **mathematical logic** that studies **sets** which, informally, are collections of objects.

Topic area: Algebra

Sub-topic area: Numbers and operations

Unit 3

Binary operations

Key unit competence

Use mathematical logic to understand and perform operations using the properties of algebraic structures.

3.0 Introductory activity

Given the operation L defined in a set containing the element a and element b such that: aLb = (a+b)÷b.

a) Verify if (aLb) belongs to the set of all real numbers when a and b are elements of this set.

b) Is the operation L commutative in the set of real numbers without zero?

3.1 Introduction

Activity 3.1

1. Discuss what you understand by the term binary operations.
2. Carry out research to find the meaning as used in mathematics.

In mathematics, a binary operation on a set is a calculation that combines two elements of the set (called **operands**) to produce another element of the set. Typical examples of binary operations are the addition (+) and multiplication (x) of numbers and matrices as well as composition of functions on a single set. For example:

- On the set of real numbers $\mathbb{R}$, f(a, b) = a ξ b is a binary operation since the sum of two real numbers is a real number.
- On the set of natural numbers $\mathbb{N}$, f(a, b) = a + b is a binary operation since the multiplication of two natural numbers is a natural number. This is a different binary operation than the previous one since the sets are different.

3.2 Groups and rings

Activity 3.2

1. Discuss what you understand by the terms binary, group, ring, integral domain and field.
2. Carry out research to find their meanings as used in mathematics.

Group

A **group** (G,*) is a non-empty set (G) on which a given binary operation (*) is defined such that the following properties are satisfied:

(a) **Closure property:** $\forall$ a, b $\in$ G, (a * b) $\in$ G.

(b) **Associative property:** $\forall$ a, b, c $\in$ G, a * (b * c) = (a* b) *c

(c) **Identity property**: $\forall$ a $\in$ G, $\exists$ e $\in$ G, a * e = e * a = a

(d) **Inverse element property :** $\forall$ a $\in$ G, $\exists$ a^{-1} $\in$ G, a * a^{-1} = a^{-1} * a = e, where e is an identity element.

Example 3.1

The set $\mathbb{N}$ of all natural numbers with the operation of addition (+), is a group. Verify this.

Solution

To verify this, check if all the four properties are satisfied.

(a) $\forall$ a, b, $\in \mathbb{N}$, (a + b) $\in \mathbb{N}$, (closure).

(b) $\forall$ a, b, c $\in \mathbb{N}$, (a + b) + c = a + (b + c), (associative).

(c) $\forall$ a $\mathbb{N}$, a + e = e + a = a. The identity element is 0.

(d) $\forall$ a $\in \mathbb{N}$, $\exists$ a^{-1} $\in \mathbb{N}$, a + a^{-1} = a + a^{-1} = 0. Since –a $\notin \mathbb{N}$, there is no inverse element .

We conclude that ($\mathbb{N}$, +) is not a group.

A group is commutative (**Abelian**) if in addition to the previous axioms, it satisfies commutativity: $\forall$ a, b $\in$ G, a * b = b * a

Example 3.2

Verify if ($\mathbb{Z}$, +) is or is not a commutative group.

Solution

($\mathbb{Z}$, +) is a commutative group if it satisfies the following:

1. **Closure:** $\forall$ a, b $\in \mathbb{Z}$, (a + b) $\in \mathbb{Z}$.
2. **Associative:** $\forall$ a, b, c $\in \mathbb{Z}$, a + (b + c) = (a + b) + c.

3. **Identity element:** $\forall a \in \mathbb{Z}, \exists e \in \mathbb{Z}, a + e = e + a = a$. The identity element is 0.
4. **Inverse element:** $\forall a \in \mathbb{Z}, \exists a^{-1} \in \mathbb{Z}, a + a^{-1} = a^{-1} + a = e$. The inverse element is $-a$.
5. **Commutativity:** $\forall a, b \in \mathbb{Z}, a + b = b + a$.

We conclude that $(\mathbb{Z}, +)$ is a commutative (Abelian) group.

Example 3.3

Verify if $(\mathbb{Z}, \bullet)$ is or is not a commutative group.

Solution

The set of integers with the operation $\bullet$ written $(\mathbb{Z}, \bullet)$ is a commutative group if it satisfies the following:

1. **Closure:** $\forall a, b \in \mathbb{Z}, (a \bullet b) \in \mathbb{Z}$.
2. **Associative:** $\forall a, b, c \in \mathbb{Z}, a \bullet (b \bullet c) = (a \bullet b) \bullet c$
3. **Identity element:** $\forall a \in \mathbb{Z}, \exists e \in \mathbb{Z}, a \bullet e = e \bullet a = a$. The identity element is 1.
4. **Inverse element:** $\forall a \in \mathbb{Z}, \exists a^{-1} \in \mathbb{Z}, a \bullet a^{-1} = a^{-1} \bullet a = e$,

   We conclude that $(\mathbb{Z}, \bullet)$ is not a commutative group. For instance the inverse of 2 is $\frac{1}{2}$ but $\frac{1}{2} \notin \mathbb{Z}$.

   We conclude that $(\mathbb{Z}, \bullet)$ is not a commutative group.

Example 3.4

Verify if the set $A = \{-2, -1, 0, 1, 2\}$ with the usual operation of addition + written $(A, +)$ is a group or not.

Solution

$(A, +)$ is a group if it satisfies the following:

1. **Closure:** $\forall a, b \in A, a + b \in A$.
2. **Associative:** $\forall a, b, c \in A, a + (b + c) = (a + b) + c$
3. **Identity element:** $\forall a \in A, \exists e \in A, a + e = e + a = a$. The identity element is 0.
4. **Inverse element:** $\forall a \in A, \exists a' \in A, a + a' = a' + a = 0$. For instance the inverse (opposite) of 1 is -1.

Thus, we conclude that $(A, +)$ is a group.

Note: The set G with a binary operation * fails to be a commutative (Abelian) group if at least one of the 5 axioms is not satisfied.

Application activity 3.1

1. Show that the set $\mathbb{R}$ is a commutative group under addition.
2. Show that the set $\mathbb{Z}$ is a group under addition.
3. Determine whether the following are groups or not.
   (a) Odd integers under addition.
   (b) The set of integers $\mathbb{Z}$ under multiplication.
   (c) The set of natural numbers $\mathbb{N}$ under addition.
   (d) The set of real numbers $\mathbb{R}$ under multiplication.
   (e) The set of rational numbers $\mathbb{Q}$ under addition.

Subgroups

A non-empty subset H of a group G is a subgroup if the elements of H form a group under the operation from G restricted to H. The entire group is a subgroup of itself and is called the **improper subgroup**.

Every group has a subgroup consisting of an identity element alone and is called the **trivial subgroup**. The identity element is an element of every subgroup of a group.

If $H \neq G$, we call it a subgroup H of G; **proper**, and we write $H < G$.

If $H \neq \{e\}$, we call it a subgroup H of G; **nontrivial**, and we write $H \leq G$.

$(H, *)$ is a subgroup of $(G, *)$ if it verifies the following conditions:

1) **Closure**: $\forall\, x, y, \in H, (x * y) \in H$
2) $e \in H$
3) $\forall\, x, \in H, x^{-1} \in H$

where e is the identity element and x^{-1} is the inverse of x.

Example 3.5

Show that a set of even numbers (call it (E, +)) is a subgroup of $(\mathbb{Z}, +)$.

Solution

We verify if (E, +) satisfies the conditions:

1) **Closure:** we know that $\forall\, x, y, \in E\ (x + y) \in E$ (the sum of even number is also even)
2) **The identity element** is $e = 0$ and $\in E$ (0 is an even number)
3) $\forall\, x, \in H, x^{-1} \in E$. (The opposite of an even number is also an even number.)

Since the all properties are verified, we conclude that the set of even numbers (E, +) is a subgroup of $(\mathbb{Z},+)$.

Other examples of other subgroups are

a) $(\mathbb{Z}, +) < (\mathbb{Q}, +) < (\mathbb{R}, +)$

b) $(\mathbb{R}_{\neq 0}, x) < (\mathbb{R}_{+}, x)$

c) $(\mathbb{Q}_{\neq 0}, x) < (\mathbb{Q}_{\neq 0}, x)$.

Rings

A set $\mathbb{R}$ with two operations + and • and defined on its domain is a ring, written $(\mathbb{R}, +, \bullet)$, if it satisfies the following axioms:

1. $(\mathbb{R}, +)$ is **commutative group**
2. **Closure** under the operation '•': $\forall$ a, b $\in \mathbb{R}$, **(a . b)** $\in \mathbb{R}$.
3. **Associative** under the operation '·': $\forall$ a, b, c $\in \mathbb{R}$, **a · (b • c) = (a • b) • c**
4. **Distributive** for '•' over '+': $\forall$ a, b, c $\in \mathbb{R}$,

   a • (b + c) = a • b + a • c and **(a + b) • c = a • c + b • c**

   A given ring $(\mathbb{R}, +, \bullet)$ is commutative if in addition the $(\mathbb{R}, \bullet)$ is

5. **Commutative:** $\forall$ a, b $\in \mathbb{R}$, then **a • b = b • a**

Example 3.6

Verify if the set of integers with ordinary addition and multiplication $(\mathbb{Z}, +, \bullet)$ is a ring or not.

Solution

1. $(\mathbb{Z},+)$ is a commutative group
2. $(\mathbb{Z},\bullet)$ is closed since: For every a, b $\in \mathbb{Z}$, then a (b • c) = (a • b) • c
3. $(\mathbb{Z},\bullet)$ is associative since: For every a, b, c $\in \mathbb{Z}$, then a. (b • c) = (a • b) ·c
4. The operation · is distributive over + : For every a, b, c $\in \mathbb{Z}$, then

   a • (b + c) = a • b + a • c and **(a + b) • c = a • c + b • c**

   We conclude that $(\mathbb{Z}, +,\bullet)$ is a ring.

Application activity 3.2

Show that in a ring $\mathbb{R}$ with a identity element 1:

(a) a • 0 = 0 • a = 0

(b) a (– b) = (–a) b = –ab

(c) (–a)(–b) = ab

(d) (–1)a = –a

(e) (–1)(–1) = 1

(f) (na) b = a(nb) = n(ab) ; na = (n1)a = a(n1)

where a, b $\in \mathbb{R}$, n $\in \mathbb{Z}$.

3.3 Fields and integral domains

Fields

A field (F,+, •) is a field if it satisfies the following conditions:

1. (F,+, •) is a ring
2. **Commutativity** under '•': For every a, b ∈ F, a • b = b • a,
3. **Identity element e** under '•': For every a ∈ F, there exists *e* ∈ F such that a • *e* = e • a = a.
4. **Inverse element** under '•': For every a ∈ F and a ≠ 0, there exists a^{-1} ∈ F such that a • a^{-1} = a^{-1}• a = *e*.

A field F is a non-empty set defined by the following properties.

For all a, b, c ∈ F:

Under + (addition property)	Under x (multiplication property)
(a) **Closure:** a + b ∈ F (b) **Associativity:** (a + b) + c = a + (b + c)	(a) **Closure:** a x b ∈ F. (b) **Associativity:** (a x b) x c = a x (b x c).
(c) **Additive inverse:** a + a^{-1} = a^{-1} + a = e for every a ∈ F, e is the identity element (d) **Additive identity:** there exists an identity element a ∈ F such that a + e = e + a = a (e) **Commutative:** a + b = b + a (f) **Distributive property:** a x (b + c) = a x b + a x c (b + c) x a = b x a + c x a	(c) **Multiplicative inverse** for every a ≠ 0, there exists an element a x a^{-1} an element a^{-1} in F such that a x a^{-1} = e, e is the identity element. (d) **Multiplicative identity**, there exists an identity element e ∈ F such that a x e = e x a = a. (e) **Commutative** a x b = b x a. (f) **Cancellation law** a ≠ e a x b = a x c implies b = c.

Integral domain

An **integral domain** is a commutative ring with an identity (1≠ 0) with no zero-divisors. That is *a*b = 0 ⇒ a = 0 or b = 0.

A ring (ℝ, +, •) is called an integral domain if:

1) (ℝ, +, •) is a commutative ring
2) (ℝ, +, •) has an identity element 1≠ 0
3) (ℝ, +, •) has no zero divisor.

Example 3.7

Show whether a ring $(\mathbb{Z}, +, \bullet)$ is an integral domain.

Solution

We know that

1) $(\mathbb{Z}, +, \bullet)$ is a commutative ring
2) $(\mathbb{Z}, +, \bullet)$ has an identity element $(1 \neq 0)$
3) $(\mathbb{Z}, +, \bullet)$ has no zero divisor.

Thus, $(\mathbb{Z}, +, \bullet)$ is the integral domain.

Note: An element $a \neq 0$ of a ring $\mathbb{R}$ is called a zero divisor if there exists $b \in \mathbb{R}$ and $b \neq 0$ such that $ab = 0$.

3.4 Cayley tables

A Cayley table is a useful device for studying binary operations on finite sets. It can be used to help determine if the set under a given operation is a group or not.

Given a binary operation (S, *), with S a finite set, its Cayley table has the elements of S listed in the top row and left hand column of the table, and inside the table we write the outcome a * b of the operation in the row labelled with a and the column labelled with b. The elements should be placed along the top row and the left side column in the same order. The following illustrates the Cayley table of the set S = {a, b} under the operation *:

*	a	b
a	a * a	a * b
b	b * a	b * b

Fig 3.1

Below we use Cayley tables to illustrate the groups of order 2, 3 and 4.

Order 2: S = {a, b}

*	a	b
a	a	b
b	b	a

Fig 3.2

We remark that a * a = a, a * b = b, b * a = b, and b * b = a

Order 3: S = {a, b, c}

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Fig 3.3

Order 4: S = {a, b, c, d}

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Fig 3.4

We can use Cayley tables to determine if (S, *) is closed, commutative, admits identity element or inverse element.

Closure: If all elements inside the table are in the original set S, then (S, *) is closure.

Commutative: The Cayley table is symmetric about the diagonal. This will only happen if every corresponding row and column are identical.

Identity: In a Cayley table, the identity element is the one that leaves all elements of the set S unchanged.

Inverse: It is found by asking: ''What other element can I combine with this one to get the identity?''

Associative: It is checked by simply respecting the rules of parentheses in comparing if the two sides are equal.

Example 3.8

The set S = {a, b, c} with binary operation * defines a group illustrated by the following Cayley table:

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Fig 3.5

(a) Is (S, *) closure?

(b) Find the identity element.

(c) Find the inverse of each element of S.

(d) Is (S, *) commutative?

(e) Is a * (b * c) = (a * b) * c?

Solution

(a) (S, *) is closure since every product inside the table is the element of S.

(b) The identity element is a since a * a = a, a * b = b * a = b, a * c = c * a = c.

(c) Since a * a = a then a^{-1} = a. Since b * c = a, then b^{-1} = c. Since c * b = a then c^{-1} = b (where a^{-1} means the inverse of a).

(d) Yes, (S, *) is commutative since a * b = b * a = b, a * c = c * a = c, and b * c = c * b = a (the products inside the table are symmetric around the main diagonal.)

(e) Yes, a * (b * c) = (a * b) * c since a * (b * c) = a * a = a = (a * b) * c = b * c = a.

Example 3.9

Verify if the set S = {a, b, c, d} with operation * represented on Cayley table of Figure 3.6 is a commutative group.

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Fig 3.6

We remark that (S, *) satisfies the following:

1. **Closure:** Each product inside the table is the element of the set S.
2. **Associative:** For instance a * (b * c) = a * d = (a * b) * c = b * c = d
3. There exists an **identity element** a : For instance a * b = b * a = b, a * c = c * a = c.
4. There exists an **inverse element**: For instance a * a = a, b * b = a, c * c = a each element is the inverse of itself.
5. **Commutative:** For instance b * c = c * b = d. It is illustrated by the products inside the table which are symmetric about the diagonal from top left to bottom right.

   By conclusion, the set S = {a, b, c, d} is a commutative group under *.

Modular arithmetic

Consider the set $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$. Let $n \in \mathbb{N}$ and $n \neq 0$. For a, b $\in \mathbb{Z}_n$ define a + b by forming the sum of a and b as integers, divide them by n and take the remainders. These operations are called the **addition modulo n**.

Example 3.10

Write down the Cayley table for addition modulo 5 on the set $\mathbb{Z}_5$ i.e. $(\mathbb{Z}_5, +_5)$ = ((mod 5), +) and verify if it is a commutative group.

Solution

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Fig 3.7

The operation + on $\mathbb{Z}_5$ is

1. **Closure:** Each product inside the table is the element of the set $\mathbb{Z}_5$.
2. **Associative:** For instance $2+(3+4)=(2+3)+4$

$$2+2 = 0+4$$
$$4 = 4$$

3. There exists an **identity element** 0: For instance $3 + 0 = 0 + 3 = 3$, $4 + 0 = 0 + 4 = 4$.
4. There exists an **inverse element**: For instance $1 + 4 = 0$, 4 is the inverse of 1. $2 + 3 = 0$, 3 is the inverse of 2. $0 + 0 = 0$, 0 is the inverse of itself.
5. **Commutative:** For instance $1 + 2 = 2 + 1 = 3$. It is illustrated by the data inside the table which are symmetric about the diagonal from top left to bottom right.

We conclude that the addition modulo 5 on the set $\mathbb{Z}_5$ is a commutative group.

Application activity 3.3

1. What property of a group is displayed in a Cayley table if:
   (a) the elements are symmetrical about the leading diagonal?
   (b) the same element does not appear more than once in any row or column?
   (c) the identity element occurs only once in each row or column?
2. Given the set $S = \{1, -1, i, -i\}$ and the binary operation '$\bullet$', where $i \bullet i = -1$, construct a Cayley table for $(S, \bullet)$ and determine whether or not the $(S, \bullet)$ is a commutative group.
3. Using a Cayley table displaying different compositions of the following functions, $f(x) = x$, $g(x) = -x$ and $h(x) = \frac{1}{x}$ on the set $S = \{f, g, h\}$,

o	*f*	*g*	*h*
f			
g			
h			

determine whether the composition on the set $S = \{f, g, h\}$

a) closure

b) associative

c) admits the identity element (and find it)

d) admits the inverse element

e) commutative.

Summary

1. A binary operation on a **set** is a calculation that combines two elements of the set (called **operands**) to produce another element of the set.
2. (G, *) is a group if it satisfies the following axioms:

   Closure: $\forall$ a, b $\in$ G, (a * b) $\in$ G.

   Associativity: $\forall$ a, b, c $\in$ G, a * (b * c) = (a* b) *c

   Existence of identity element: $\forall$ a $\in$ G, $\exists$ e $\in$ G, a * e = e * a = a

   Existence of inverse element: $\forall$ a $\in$ G, $\exists$ a^{-1} $\in$ G, a * a^{-1} = a^{-1} * a = e, where e is an identity element
3. A non-empty subset H of a group G is a subgroup if the elements of H form a group under the operation from G restricted to H.
4. A set ($\mathbb{R}$, +, •) is a ring if it satisfies the following axioms:

   ($\mathbb{R}$, +) is a commutative group

   Closure under the operation '•': $\forall$ a, b $\in$ $\mathbb{R}$, (**a • b**) $\in$ $\mathbb{R}$.

   Associative under the operation '•': $\forall$ a, b, c $\in$ $\mathbb{R}$, **a • (b • c) = (a • b) • c**

   Distributive for '•' over **'+':** $\forall$ a, b, c $\in$ $\mathbb{R}$,

   a • (b + c) = a • b + a · c and **(a + b) • c = a • c + b • c**

   A given ring ($\mathbb{R}$, +, •) is commutative if in addition the ($\mathbb{R}$, ·) is

   Commutative: $\forall$ a, b $\in$ $\mathbb{R}$, then **a • b = b • a**
5. A field consists of a set F and two binary operations '+' and '•', defined on $\mathbb{R}$, for which the following conditions are satisfied:

   (F, +, •) is a ring

   Commutativity under '•' For every a, b $\in$ F, a • b = b • a

Topic area: Algebra

Sub-topic area: Numbers and operations

Unit 4

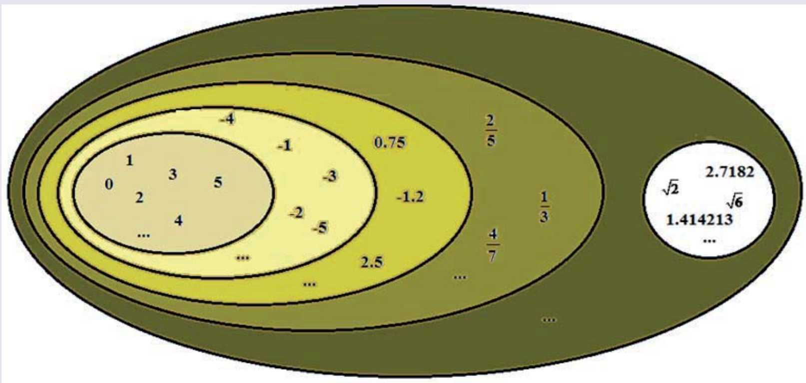
Set ℝ of real numbers

Key unit competence

Think critically using mathematical logic to understand and perform operations on the set of real numbers and its subsets using the properties of algebraic structures.

4.0 Introductory activity

From the following diagram, discuss and work out the given tasks:



1. How many sets of numbers do you know? List them down and give reasons for your answer.
2. Using a mathematical dictionary or the internet, define the sets of numbers you listed in (1).
3. Give an example of element for each set of numbers you listed.
4. Establish the relationship between the set of numbers that you listed.

4.1 Introduction

Activity 4.1

1. Discuss the following: What are sets? What are real numbers?
2. Carry out research on sets of numbers to determine the meanings of natural numbers, integers, rational numbers and irrational numbers.

The rational and irrational numbers together make up the set of real numbers denoted by $\mathbb{R}$. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all subsets of $\mathbb{R}$. In fact $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

The one-to-one correspondence between the real numbers and the points on the number line is familiar to us all. Corresponding to each real number there is exactly one point on the line: corresponding to each point on the line there is exactly one real number.

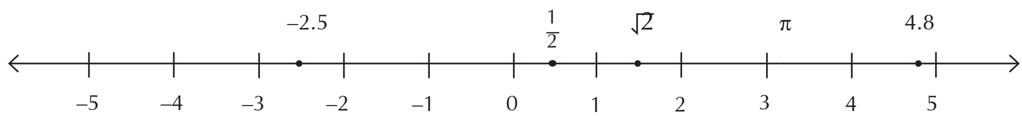


Fig 4.1

Throughout this book the following notation for various subsets of $\mathbb{R}$ will be used. In case a and b are real numbers with $a < b$.

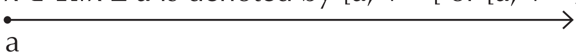
$\mathbb{R} =]-\infty, +\infty[$ or $\mathbb{R} = (-\infty, +\infty)$

$x \in \mathbb{R} | x < a$ is denoted by $]-\infty, a[$ or $(-\infty, a)$

$x \in \mathbb{R} | x \leq a$ is denoted by $]-\infty, a]$ or $(-\infty, a]$

$x \in \mathbb{R} | x > a$ is denoted by $]a, +\infty[$ or $(a, +\infty)$

$x \in \mathbb{R} | x \geq a$ is denoted by $[a, +\infty[$ or $[a, +\infty)$



$x \in \mathbb{R} | a < x < b$ is denoted by $]a,b[$ or (a, b)

$x \in \mathbb{R} | a \leq x \leq b$ is denoted by $[a,b]$

$x \in \mathbb{R} | a < x \leq b$ is denoted by $]a,b]$ or $(a, b]$

$x \in \mathbb{R} | a \leq x < b$ is denoted by $[a,b[$ or $[a, b)$

4.2 Properties of real numbers

Subsets of real numbers

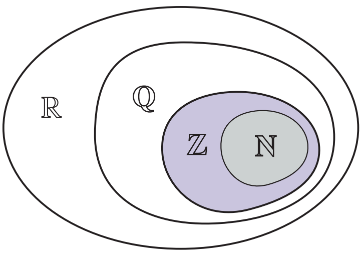
The natural numbers are $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

The integers are $\mathbb{Z} = \{ ..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

The rational numbers are $\mathbb{Q} = \{\frac{p}{q}: p, q, \in \mathbb{Z}, q \neq 0\}$.

A rational number is one which can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

The numbers $\frac{3}{4}, -\frac{2}{3}, 6 = \frac{6}{1}, 0.\overline{3} = \frac{1}{3}, \sqrt{4} = \frac{2}{1}$ are all rational, whereas the numbers $\sqrt{2}, \frac{2}{\sqrt{3}}, 4\sqrt{5} - 6, \pi$ are not rational. They are said to be irrational.



Natural numbers form a subset of integers. Integers form a subset of rational numbers. Rational numbers are a subset of real numbers.

All rational numbers can be expressed as either finite or recurring decimals. For example, $\frac{1}{2} = 0.5$ is a finite decimal and $\frac{1}{3} = 0.\overline{3}$ is a recurring decimal. An irrational number cannot be expressed in this way.

Example 4.1

Express the recurring decimal $0.2\overline{45}$ in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Solution

Let $x = 0.2\overline{45}$.

Then $100x = 24.5\overline{45}$.

By subtraction we obtain $99x = 24.3$. Hence $0.2\overline{45} = \frac{243}{990} = \frac{27}{110}$.

Activity 4.2

Use a marker and manila paper to draw a larger copy of the table below.

Identify the sets to which each of the following numbers belong by marking an "X" in the appropriate boxes.

	Number	Natural numbers	Whole numbers	Integers	Rational numbers	Irrational numbers	Real numbers
1.	–17						
2.	–2						
3.	$-\frac{9}{37}$						
4.	0						
5.	–6.06						
6.	$4.5\overline{6}$						
7.	3.050050005...						
8.	18						
9.	$\frac{-43}{0}$						
10.	π						

Order properties

If a and b are any two real numbers, then either $a < b$ or $b < a$ or $a = b$. The sum and product of any two positive real numbers are both positive.

If $a < 0$ then $(-a) > 0$.

If $a < b$ then $(a - b) > 0$. This means that the point on the number line corresponding to a is to the right of the point corresponding to b.

The elementary rules for inequalities are:

1. If $a < b$ and c is any real number then $a + c < b + c$. That is we may add (or subtract) any real number to (or from) both sides of an inequality.
2. (a) If $a < b$ and $c > 0$, then $ac < bc$.

   (b) If $a < b$ and $c < 0$, then $ac > bc$.

   We may multiply (or divide) both sides of an inequality by a positive real number, but when we multiply (or divide) both sides of an inequality by a negative real number we must change the direction of the inequality sign.
3. (a) If $0 < a < b$ then $0 < a^2 < b^2$.

   (b) If $a < b < 0$ then $a^2 > b^2$.

   We may square both sides of an inequality if both sides are positive. However if both sides are negative we may square both sides but we must reverse the direction of the inequality sign and then both sides become positive. If one side is positive and the other negative we cannot use the same rule. For example, $-2 < 3$ and $(-2)^2 < 3^2$, but $-3 < 2$ and $(-3)^2 > 2^2$.

4. (a) If $0 < a < b$ then $0 < \frac{1}{b} < \frac{1}{a}$.

(b) If $a < b < 0$ then $\frac{1}{b} < \frac{1}{a} < 0$.

That is we may take the reciprocal of both sides of an inequality only if both sides have the same sign and, in each possible case, we must reverse the direction of the inequality sign.

4.3 Absolute value functions

Activity 4.3

Carry out a research on the meaning of absolute value. Present your findings to the rest of the class for discussion. Use suitable examples.

There is a technical definition for **absolute value**, but you could easily never need it. For now, you should view the absolute value of a number as its distance from zero.

Let us look at the number line.

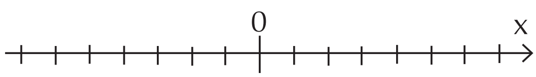


Fig 4.2

The absolute value of x, denoted "| x |" (and which is read as "the absolute value of x"), is the distance of x from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?" This means not only that | 3 | = 3, because 3 is three units to the right of zero, but also that | –3 | = 3, because –3 is three units to the left of zero.

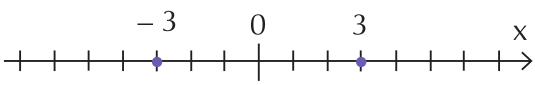


Fig 4.3

Note: The absolute-value notation is bars, not parentheses or brackets. Use the proper notation. The other notations do not mean the same thing.

Example 4.2

Simplify –| –3 |.

Solution

Given –| –3 |, first handle the absolute value part, taking the positive and converting the absolute value bars to parentheses:

$-|-3| = -(+3)$

Now we can take the negative through the parentheses. So,

$-|-3| = -(3) = -3$

As this illustrates, if you take the negative of an absolute value, you will get a negative number for your answer.

Sometimes we order numbers according to their size. We denote the size or absolute value of the real numbers x by $|x|$, called the **modulus of** x.

Definition $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$.

Since the symbol $\sqrt{\ }$ is used to mean any non-negative square root, we may also define the modulus of the real number x by:

$|x| = \sqrt{x^2}$

In general, if a is any positive number:

$|x| < a \Rightarrow -a < x < a$ and $|x| > a \Rightarrow x > a$ or $x < -a$.

The following relations are true for all real numbers a and b:

(1) $|-a| = |a|$

(2) $-|a| \leq a \leq |a|$

(3) $|ab| = |a||b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $(b \neq 0)$

(4) $|a + b| \leq |a| + |b|$

(5) $|a - b| \geq |a| - |b|$

Example 4.3

Solve the inequality $|3x - 2| < 1$.

Solution

$|3x - 2| < 1$

$-1 < 3x - 2 < 1$

$1 < 3x < 3$

$\frac{1}{3} < x < 1$.

Application activity 4.1

Solve the following:

a) $|5 - x| = \frac{5}{2}$ b) $|x| \leq 3$ c) $|2x + 7| \geq 3$

d) $|3 - 2x| = 5$ e) $|x + 2| > 1$ f) $|4x - 1| < 15$

g) $|9 - 3x| \geq 0$ h) $|2x + 3| \geq x - 4$

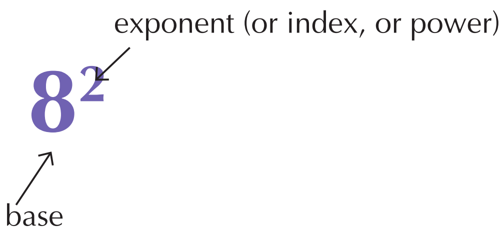
4.4 Powers and radicals

Powers

Activity 4.4

1. Revise the meaning of the term 'power'. Give the definition.
2. Research and derive the rules of exponential expressions.

The power of a number says how many times we use the number in a multiplication. It is written as a small number to the right and above the base number. Another name for power is **index** or **exponent**.



In the above example: $8^2 = 8 \times 8 = 64$

When a real number a is raised to the index m to give a^m , the result is a power of a. When a^m is formed, m is sometimes called the power, but it is more correctly called the index to which a is raised.

The rules used to manipulate exponential expressions should already be familiar with the reader. These rules may be summarized as follows:

1. Product Rule: $(a^m)(a^n) = a^{m+n}$
2. Quotient Rule: $\frac{a^m}{a^n} = a^{m-n}$, $(a \neq 0)$
3. Power of Power Rule: $(a^m)^n = a^{mn}$
4. Power of Product Rule: $(ab)^m = a^m b^m$
5. Power of a Quotient Rule: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $(b \neq 0)$.

For the Quotient Rule to hold when m = n we must agree that,

$\frac{a^m}{a^m} = a^{m-m} = a^0$, $(a \neq 0)$. Then it is natural to define $a^0 = 1$, $(a \neq 0)$.

Now $\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$, $(a \neq 0)$, which gives the meaning to negative exponents.

Rational exponents

If the product rule for exponents is to hold, then $\left(a^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}}\right) = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$. But we know that $(\sqrt{a})(\sqrt{a}) = a$ $(a \geq 0)$. Thus $a^{\frac{1}{2}}$ is a square root of a (provided $a \geq 0$). To avoid any confusion, take $a^{\frac{1}{2}} = \sqrt{a}$ $(a \geq 0)$, i.e., $a^{\frac{1}{2}}$ is taken to be the positive square root of a (a > 0).

More generally, if n is a positive integer, $a^{\frac{1}{n}}$ is defined by $a^{\frac{1}{n}} = \sqrt[n]{a}$.
Note that if n is even, a must be non-negative; if n is odd, a may be any real number.

Next we define $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$ which exists for all a if n is odd and exists for $a \geq 0$ if n is even.

Example 4.4

Solve the following equations:

(a) $8^{1-x} = 4^{2x+3}$ (b) $a^{\frac{3}{2}} = 8$ (c) $a^{\frac{2}{3}} = 16$.

Solution

(a) $8^{1-x} = 4^{2x+3}$ Here we may express each side as a power of 2.

$(2^3)^{1-x} = (2^2)^{2x+3}$

$2^{3-3x} = 2^{4x+6}$

$3 - 3x = 4x + 6$

$7x = -3$

$x = -\frac{3}{7}$

(b) $a^{\frac{3}{2}} = 8$

$a^{\frac{1}{2}} = \sqrt[3]{8} = 2$

$a = 4$

(c) $a^{\frac{2}{3}} = 16$

$a^{\frac{1}{3}} = \pm 4$

$a = \pm 64$

Example 4.5

Evaluate the following:

a) $(2^3)(2^4)$ b) $\frac{7^8}{7^5}$

c) $\left(2^3\right)^4$

Solution

a) $(2^3)(2^4) = 2^{3+4} = 2^7 = 128$

b) $\frac{7^8}{7^5} = 7^{8-5} = 7^3 = 343$

c) $\left(2^3\right)^4 = 2^{12} = 4096$

Example 4.6

Evaluate the following:

a) $\left(3x^2\right) = 36$ b) $\left(\frac{x}{5}\right)^2 = \frac{1}{625}$ c) $\frac{1}{2^{3x}} = 8$

Solution

a)
$$\left(3x\right)^2 = 36$$
$$9x^2 = 36$$
$$x^2 = 4$$
$$|x|^2 = 2^2$$
$$|x| = 2$$
$$\pm x = 2$$
$$x = \pm 2$$

b)
$$\left(\frac{x}{5}\right)^2 = \frac{1}{625}$$
$$\frac{x^2}{25} = \frac{1}{625}$$
$$625x^2 = 25$$
$$x^2 = \frac{1}{25}$$
$$|x|^2 = \left(\frac{1}{5}\right)^2$$
$$|x| = \frac{1}{5}$$
$$\pm x = \frac{1}{5}$$
$$x = \pm\frac{1}{5}$$

c)
$$\frac{1}{2^{3x}} = 8$$
$$2^{-3x} = 2^3$$
$$-3x = 3$$
$$x = -1$$

Example 4.7

Simplify the following

a) $\sqrt[3]{27}$

b) $\sqrt[4]{64}$

c) $(\sqrt{8})^6$

Solution

a) $\sqrt[3]{27} = \sqrt[3]{3^3} = 3^{\frac{3}{3}} = 3^1 = 3$

b) $\sqrt[4]{64} = \sqrt[4]{2^6} = 2^{\frac{6}{4}} = 2^{\frac{3}{2}} = \sqrt{2^3} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

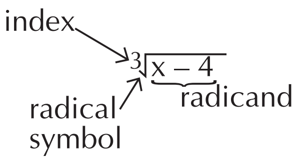
c) $(\sqrt{8})^6 = (8^{\frac{1}{2}})^6 = 8^{\frac{6}{2}} = 8^3 = 512$

Radicals

Activity 4.5

Research on the meaning of the term radical. Discuss your findings with the rest of the class. Your teacher will assist you get a concise meaning of the term.

A radical is an expression that has a root, such as square root, cube root, etc. The symbol is $\sqrt{\ }$.



Cube root of "x – 4"

The symbol $\sqrt{\ }$ used in connection with square roots, cube roots and n^{th} roots, for larger values of n. The notation $\sqrt{a}$ indicates square root, $\sqrt[3]{a}$ indicates cube root and $\sqrt[n]{a}$ indicates n^{th} root of a.

Square roots

A square root of a real number a is a number x such that $x^2 = a$. If a is negative, there is no such real number. If a is positive, there are two such numbers, one positive and one negative. For $a \geq 0$, the notation $\sqrt{a}$ is used to denote quite specifically the non-negative square root of a.

1. The symbol $\sqrt{\ }$ denotes the positive square root only.
2. Every positive number has two square roots, one of which is positive and the other negative.
3. The square root of zero is zero.
4. Negative real numbers have no real square roots.

Cube roots

If $a^3 = b$, then b is the cube of a and a is the cube root of b. This time we say the cube root of b since $2^3 = 8$ but $(–2)^3 = –8$.

1. No definition is needed to specify which cube root is meant since a real number has one and only one cube root in the real number system.
2. Every positive number has one and only one cube root which is positive.
3. The cube root of zero is zero.
4. Every negative number has one and only one cube root which is negative.

N^{th} roots

In general, if n is an even positive integer then in the real number system:

1. If $a > 0$, a has two n^{th} roots, one positive and another negative.
2. If $a = 0$, a has one n^{th} root which is zero.
3. If $a < 0$, a has no real n^{th} roots.

If n is an odd positive integer (not 1) then in the real number system,

1. If $a > 0$, a has one n^{th} root which is positive.
2. If $a = 0$, a has one n^{th} root which is zero.
3. If $a < 0$, a has one n^{th} root which is negative.

If n is a positive integer other than 1 or 2 the radical sign $\sqrt[n]{}$ is used to indicate an n^{th} root of a real number. If $n = 2$ just the radical $\sqrt{}$ is used.

Surds

A surd is an irrational number expressed with a radical (root) sign.

$\sqrt{2}$, $2 + \sqrt{7}$, $\sqrt[3]{4}$, $\frac{3}{\sqrt{3}+1}$ are true surds.

$\sqrt{4}$, $\sqrt{6\frac{1}{4}}$, $1 + \sqrt[3]{8}$, $\sqrt{0.64}$, $\frac{7}{\sqrt[4]{16}}$ are written in surd form but are not true surds since they can be written without the radical as 2, $\frac{5}{2}$, 3, 0.8, and 3.5 respectively.

The rules used to manipulate surds may be summarised as follows:

1. Product Rule: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for $a \geq 0, b \geq 0$,
2. Converse of Product Rule: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ for $a \geq 0, b \geq 0$,
3. Quotient Rule: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ for $a \geq 0, b > 0$

Note: If a, b, c and d are rational with b and d not both squares of rational numbers, then $a\sqrt{b} + c\sqrt{d}$ is a **binomial** surd. The **conjugate** of the binomial surd $a\sqrt{b} + c\sqrt{d}$ is $a\sqrt{b} - c\sqrt{d}$.

Note that the product of a binomial surd and its conjugate is rational:

$(a\sqrt{b} + c\sqrt{d})(a\sqrt{b} - c\sqrt{d}) = a^2b - c^2d$

We often use this result to simplify surd expressions, particularly the quotient of binomial surds.

Rationalization

If a simple surd such as $\sqrt{3}$ is multiplied by itself, the result is rational: $\sqrt{3} \times \sqrt{3} = 3$. If a binomial surd is multiplied by its conjugate, the result is also a rational.

We make use of these results when simplifying certain surd expressions by '**rationalizing the denominator**'.

Example 4.8

Express $\frac{\sqrt{2}}{\sqrt{5}}$ with a rational denominator.

Solution

$$\frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{5}$$

Example 4.9

Express $\frac{3}{3\sqrt{2} - 2\sqrt{3}}$ with a rational denominator and in its simplest form.

Solution

$$\frac{3}{3\sqrt{2} - 2\sqrt{3}} = \frac{3}{3\sqrt{2} - 2\sqrt{3}} \times \frac{3\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}} = \frac{3(3\sqrt{2} + 2\sqrt{3})}{18 - 12} = \frac{3(3\sqrt{2} + 2\sqrt{3})}{6} = \frac{3\sqrt{2} + 2\sqrt{3})}{2}$$

Application activity 4.2

Simplify and give the answer in irrational form

1. $\frac{3}{\sqrt{2}}$
2. $\frac{1}{\sqrt{7}}$
3. $\frac{2}{\sqrt{11}}$
4. $\frac{3\sqrt{2}}{\sqrt{5}}$
5. $\frac{1}{\sqrt{27}}$
6. $\frac{\sqrt{5}}{\sqrt{10}}$
7. $\frac{1}{\sqrt{2} - 1}$
8. $\frac{3\sqrt{3}}{5 + \sqrt{2}}$
9. $\frac{2}{2\sqrt{3} - 3}$
10. $\frac{5}{2 - \sqrt{5}}$
11. $\frac{1}{\sqrt{7} - \sqrt{3}}$
12. $\frac{4\sqrt{3}}{2\sqrt{3} - \sqrt{3}}$
13. $\frac{3 - \sqrt{5}}{\sqrt{5} + 1}$
14. $\frac{2\sqrt{3} - 1}{4 - \sqrt{3}}$
15. $\frac{\sqrt{5} - 1}{\sqrt{5} - 2}$
16. $\frac{3}{\sqrt{3} - \sqrt{2}}$
17. $\frac{3\sqrt{5}}{2\sqrt{5} + 1}$
18. $\frac{\sqrt{2} + 1}{\sqrt{2} - 1}$
19. $\frac{2\sqrt{7}}{\sqrt{7} + 2}$
20. $\frac{\sqrt{5} - 1}{3 - \sqrt{5}}$
21. $\frac{1}{\sqrt{11} - \sqrt{7}}$
22. $\frac{4 - \sqrt{3}}{3 - \sqrt{3}}$
23. $\frac{1 - 3\sqrt{2}}{3\sqrt{2} + 2}$
24. $\frac{1}{3\sqrt{2} - 2\sqrt{3}}$
25. $\frac{\sqrt{3}}{\sqrt{2}(\sqrt{6} - \sqrt{3})}$

Irrational equations

The process of squaring both sides is used in solving irrational equations, i.e., equations involving surd expressions. It is therefore always necessary to check any solution obtained.

Example 4.10

Solve the equation $\sqrt{3x + 4} - \sqrt{8 - x} = 2$.

Solution

Transpose one surd: $\sqrt{3x + 4} = 2 + \sqrt{8 - x}$

Square both sides: $3x + 4 = 4 + 4\sqrt{8 - x} + 8 - x$

$4x - 8 = 4\sqrt{8 - x}$

$x - 2 = \sqrt{8 - x}$

Square both sides again: $x^2 - 4x + 4 = 8 - x$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

Thus, $x = 4$ or $x = -1$.

Check:

If $x = 4$,4 $\sqrt{3(4) + 4} - \sqrt{8 - 4} = \sqrt{16} - \sqrt{4} = 4 - 2 = 2$, which checks.

If $x = -1$, $\sqrt{3(-1) + 4} - \sqrt{8 - (-1)} = \sqrt{1}, -\sqrt{9} = 1 - 3 = -2$, which does not check.

Therefore $x = 4$ is the only solution.

4.5 Decimal logarithms

Activity 4.6

Discuss in groups of four and give two examples for each.

1. What is a decimal?
2. What is a logarithm?

Many positive numbers can be easily written in the form 10^x. For example

$1,000 = 10^3$

$100 = 10^2$

$10 = 10^1$

$1 = 10^0$

$0.1 = 10^{-1}$

$0.01 = 10^{-2}$

$0.001 = 10^{-3}$, and so on.

Also, numbers like $\sqrt{10}$, $\sqrt[3]{10}$, $10\sqrt{10}$, $\frac{1}{\sqrt{10}}$ and $\frac{1}{\sqrt[3]{10}}$ can be written in the form 10^x.

$\sqrt{10} = 10^{\frac{1}{2}}$

$\sqrt[3]{10} = 10^{\frac{1}{3}}$

$10\sqrt{10} = 10^1 \times 10^{\frac{1}{2}} = 10^{1+\frac{1}{3}} = 10^{\frac{3}{2}}$

$\frac{1}{\sqrt{10}} = 10^{-\frac{1}{2}}$

$\frac{1}{\sqrt[3]{10}} = 10^{-\frac{1}{3}}$

In fact, all positive numbers can be written in the form 10^x by introducing the concept of logarithms.

Definition

The logarithm of a positive number, in base 10, is its power of 10. For instance:

$100 = 10^2$, we write $\log_{10} 100 = 2$ or $\log 100 = 2$

$0.001 = 10^{-3}$, we write $\log_{10}(0.001) = -3$ or $\log(0.001) = -3$

In algebraic form, $A = 10^{\log A}$ for any $A > 0$.

Notice also that $\log 1{,}000 = \log 10^3 = 3$ and $\log 0.01 = \log 10^{-2} = -2$; this means that $\log 10^x = x$

Note: If no base is indicated we assume that it is base 10.

Example 4.11

Without using a calculator, find:

(a) $\log 1{,}000$ (b) $\log \sqrt{10}$ (c) $\log(\sqrt[3]{10})$

Solution

(a) $\log 1000 = \log 10^3 = 3$

(b) $\log \sqrt{10} = \log 10^{\frac{1}{2}} = \frac{1}{2}$

(c) $\log(\sqrt[3]{10}) = \log 10^{\frac{1}{3}} = \frac{1}{3}$

Application activity 4.3

Without using a calculator, find:

(a) $\log 1{,}000{,}000$	(b) $\log 10$	(c) $\log 0.001$
(d) $\log 1$	(e) $\log \sqrt{10}$	(f) $\log (\sqrt[4]{10})$
(g) $\log \left(\frac{1}{\sqrt[4]{10}}\right)$	(h) $\log 100\sqrt{10}$	(i) $\log \sqrt[3]{100}$
(j) $\log\left(\frac{10}{\sqrt{10}}\right)$	(k) $\log (1000\sqrt[3]{10})$	(l) $\log 10\sqrt{10}$
(m) $\log 10^{2a}$	(n) $\log (10^a \times 1{,}000)$	(o) $\log \left(\frac{10^a}{10}\right)$
(p) $\log \left(\frac{10^a}{10^b}\right)$		

The laws of logarithms

Activity 4.7

Your teacher will guide you in deriving the laws of logarithms. Use numerous examples to practise using them.

1. $\text{Log}\ (AB) = \log A + \log B$
2. $\text{Log}\ \left(\frac{A}{B}\right) = \log A - \log B,\ B \neq 0$
3. $\text{Log}\ (A^n) = n \log A$

These laws are easily established using index laws; they correspond to the first three index laws.

Since $A = 10^{\log A}$ and $B = 10^{\log B}$

1. $AB = 10^{\log A} \times 10^{\log B} = 10^{\log A + \log B}$

   However, $AB = 10^{\log AB}$

   Therefore, $\log A + \log B = \log (AB)$

2. $A^n = (10^{\log A})^n = 10^{n\log A}$

   However, $A^n = 10^{\log (A^n)}$.

Therefore, $n \log A = \log (A^n)$.

3. $\frac{A}{B} = \frac{10^{\log A}}{10^{\log B}} = 10^{\log \left(\frac{A}{B}\right)}$

Example 4.12

Write as a single logarithm in the form log a, a ∈ ℝ,

a) 3 log 5 – 2 log 4 b) 3 log 4 + 6

Solution

a) $3 \log 5 - 2 \log 4 = \log (5^3) - \log (4^2) = \log 125 - \log 16 = \log (\frac{125}{16})$

b) $3 \log 4 + 6 = \log (4^3) + 6 \log 10 = \log 64 + \log (10^6) = \log 64 + \log 1{,}000{,}000$

$= \log (64 \times 1{,}000{,}000) = \log 64{,}000{,}000$

Example 4.13

Simplify $\frac{\log 4}{\log 2}$ without using a calculator.

Solution

$$\frac{\log 4}{\log 2} = \frac{\log 2^2}{\log 2} = \frac{2 \log 2}{\log 2} = 2$$

Example 4.14

Prove the following:

(a) $\log \left(\frac{1}{32}\right) = -5 \log 2$

(b) $\log 2{,}000 = 4 - \log 5$

Solution

a) $\log (\frac{1}{32}) = \log (2^{-5}) = -5 \log 2$

b) $\log 2{,}000 = \log (\frac{10{,}000}{5}) = \log 10{,}000 - \log 5 = \log 10^4 - \log 5 = 4 - \log 5$

The logarithm can be used in solving exponential equations.

Example 4.15

Solve the following equation.

$2^{3x} = 3^{2x-1}$

Solution

Since $2^{3x} = 3^{2x-1}$, then $\log 2^{3x} = \log 3^{2x-1}$

$3x \log 2 = (2x - 1) \log 3$

$3x \log 2 = 2x \log 3 - \log 3$

$2x \log 3 - 3x \log 2 = \log 3$

$x(2\log 3 - 3 \log 2) = \log 3$

$x = \frac{\log 3}{2 \log 3 - 3 \log 2} = 9.33$

Application activity 4.4

1. Express each of the following as a single logarithm:
   (a) $\log 6 + \log 5$ (b) $\log 16 - \log 2$
   (c) $\log 24 - \log 6$ (d) $\log 7 + \log 5$
   (e) $\log 8 + \log \frac{1}{2}$ (f) $\log 5 + \log 3 + \log 7$
   (g) $\log 9 + 1$ (h) $\log 20 - 1$
   (i) $\log 8 + \log 6 - \log 3$ (j) $\log 6 + 3$
   (k) $\log 800 - 2$ (l) $\log 56 - \log 7 - \log 4$
2. Simplify by writing each of the following as a single logarithm or write as an integer:
   (a) $3 \log 2 + \log 4$ (b) $3 \log 2 + 5 \log 2$
   (c) $2 \log 10 + \log 5$ (d) $2 \log 10 - 2 \log 2$
   (e) $\frac{1}{3} \log 8 + \log 7$ (f) $\frac{1}{2} \log 36 + \log \frac{1}{3}$
   (g) $4 \log 3 - 2\log 9 + 1$ (h) $3 - 2 \log 3 + \log 8$
   (i) $2 - \frac{1}{2} \log 4 - \log 5$
3. Simplify without using a calculator:
   (a) $\frac{\log 9}{\log 3}$ (b) $\frac{\log 64}{\log 16}$ (c) $\frac{\log 100}{\log \sqrt{10}}$
   (d) $\frac{\log 9}{\log 27}$ (e) $\frac{\log 32}{\log \frac{1}{4}}$ (f) $\frac{\log \frac{1}{27}}{\log 81}$

   You can then check your answers using a calculator.
4. Show that:
   (a) $\log 9 = 2 \log 3$ (b) $\log \sqrt{2} = \frac{1}{2} \log 2$
   (c) $\log (\frac{1}{8}) = -3 \log 2$ (d) $\log (\frac{1}{5}) = - \log 5$
   (e) $\log 5 = 1 - \log 2$ (f) $\log 5{,}000 = 4 - \log 2$

Application of exponents in real life

Compound interest

The formula for interest that is compounded is

$A = P\left(1+\frac{r}{n}\right)^{nt}$, where

A represents the amount of money after a certain amount of time

P represents the principle or the amount of money you start with

r represents the interest rate and is always represented as a decimal

n is the number of times interest is compounded in one year

- if interest is compounded annually then n = 1
- if interest is compounded quarterly then n = 4
- if interest is compounded monthly then n = 12

t represents the amount of time in years.

Example 4.16

Suppose your parents invest 10,000 FRW in a savings account for college at the time you are born. The average interest rate is 4% and is compounded quarterly. How much money will be in the college account when you are 18 years old?

We will use our formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ and

let P = 10,000, r = 0.04, n = 4 and t = 18.

$A = 10{,}000\left(1+\frac{0.04}{4}\right)^{4(18)} = 20{,}471$

Suppose your parents had invested that same 10,000 FRW in a money market account that averages 8% interest compounded monthly. How much would you have for college after 18 years?

P = 10,000, r = 0.08, n = 12 and t = 18

$A = 10{,}000\left(1+\frac{0.08}{12}\right)^{12(18)} = 42{,}005.7$

Application activity 4.5

1. The population of a town in 2013 was estimated to be 35,000 people. It has an annual rate of increase (growth) of about 2.4%.
   a) What is the growth factor for the town?
   b) Write an equation to represent future growth.
   c) Use your equation to estimate the population in 2007 to the nearest hundred people.
2. Mutesi invests 300,000 FRW at a bank that offers 5% compounded annually.
   a) What is the growth factor for the investment?

b) Write an equation to model the growth of the investment.

c) How many years will it take for the initial investment to double?

3. Murerwa bought a new car at a cost of 17, 850,000 FRW. The car depreciates by approximately 15% of its value each year.

   a) What is the depreciation factor for the value of this car?

   b) Write an equation to model the depreciation value of this car.

   c) What will the car be worth in 10 years?

4. Jose invests 500,000 FRW at a bank offering 10% compounded quarterly. Find the amount of the investment at the end of 5 years (if untouched).

5. A certain strain of bacteria that is growing on your kitchen counter doubles every 5 minutes. Assuming that you start with only one bacterium, how many bacteria could be present at the end of 96 minutes?

Summary

1. The rational and irrational numbers together make up the set of **real numbers** denoted by $\mathbb{R}$. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all subsets of $\mathbb{R}$.

2. A **rational** number is one which can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

3. If a and b are any two real numbers, then either $a < b$ or $b < a$ or $a = b$.

4. The sum and product of any two positive real numbers are both positive.

5. We denote the size or **absolute value** of the real numbers x by $|x|$, called the **modulus** of x.

6. When a real number a is raised to the **index** m to give a^m, the result is a **power** of a.

7. The sign $\sqrt{\ }$ is used in connection with square roots, cube roots and n^{th} roots, for larger values of n.

8. A **surd** is an irrational number expressed with a **radical** (root) sign.

9. If a, b, c and d are rational with both b and d not squares of rational numbers, then $a\sqrt{b} + c\sqrt{d}$ is a binomial surd.

Topic Area: Algebra

Sub-topic Area: Equations and inequalities

Unit 5

Linear equations and inequalities

Key unit competence

Model and solve algebraically or graphically daily life problems using linear equations or inequalities.

5.0 Introductory activity

If x is the number of pens for a learner, the teacher decides to give everyone two more pens. What is the number of pens that a learner with one pen will have?

a) Complete the following table called table of value to indicate the number $y=f(x)=x+2$ of pens for a learner who had x pens for $x\geq 0$.

x	-2	-1	0	1	2	3
$y=f(x)=x+2$			2			
(x,y)			(0.2)			

b) Use the coordinates of points obtained in the table and complete them in the Cartesian plan.

c) Join all points obtained. What is the form of the graph obtained?

d) Suppose that instead of writing $f(x)=x+2$ you write the equation $y=x+2$. Is this equation a linear equation or a quadratic equation? What is the type of the inequality " $x+2\geq 0$"?

3. Find out an example of problem from the real life situation that can be solved by the use of linear equation in one unknown.

5.1 Equations and inequalities in one unknown

In Junior Secondary, we learnt about linear equations and inequalities.

Activity 5.1

Research on the following:

1. What is a linear equation?
2. What is a linear inequality?

Present your findings, with clear examples, to the rest of the class.

Linear equations

A linear equation is an equation of a straight line. For example:

$y = 2x + 1$

$5x = 6 + 3y$

$\frac{y}{2} = 3 - x$

Let us look more closely at one example:

$y = 2x + 1$ is a linear equation.

The graph of $\mathbf{y = 2x+1}$ is a straight line.

- When x increases, y increases **twice as much**, hence 2x
- When x is 0, y is already 1. Hence +1 is also needed
- So, $y = 2x + 1$

The general form of the linear equation with one variable is $\mathbf{ax + b = 0}$

Where $a, b \in \mathbb{R}$: and $a \neq 0$ the value of x in which the equality is verified is called the **root**. (solution of the equation).

Solving linear equations

A linear equation is a polynomial of degree 1.

In order to solve for the unknown variable, you must isolate the variable.

In the order of operation, multiplication and division are completed before addition and subtraction.

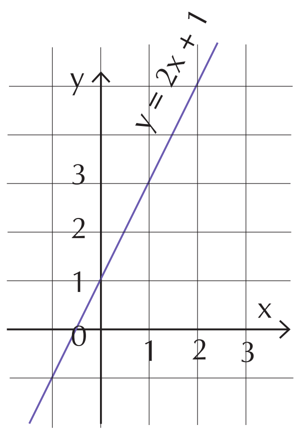


Fig 5.1

Example 5.1

Find x if $2x + 4 = 10$.

Solution

1. Isolate "x" to one side of the equation by subtracting 4 from both sides:

$$2x + 4 - 4 = 10 - 4$$
$$2x = 6$$

2. Divide both sides by 2:

$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$

3. Check your work with the original equation:

$$2x + 4 = 10$$
$$(2 \quad 3) + 4 = 10$$
$$6 + 4 = 10.$$

Example 5.2

Find x if $3x - 4 = -10$.

Solution

1. Isolate "x" to one side of the equation by adding 4 to both sides:

$$3x - 4 + 4 = -10 + 4$$
$$3x = -6$$

2. Divide both sides by 3:

$$\frac{3x}{3} = -\frac{6}{3}$$
$$x = -2$$

3. Check your work with the original equation:

$$(3 \quad -2) - 4 = -10$$
$$-6 - 4 = -10.$$

Example 5.3

Activity 5.2

Discuss in groups and verify that the following are true.

Case 1: $a \neq 0$ and $b \neq 0$, the equation has one root (solution)

$x = -\frac{b}{a}$.

Case 2: $a \neq 0$ and $b = 0$, the equation becomes $ax = 0$ and $x = 0$ since $a \neq 0$.

Case 3: $a = 0$ and $b \neq 0$, the equation becomes $0x = -b$ since there is no real number x and $b \neq 0$ such that $0x = -b$; $x = \frac{-b}{0}$ and thus the given equation is an impossible equation.

Case 4: $a = 0$ and $b = 0$, the equation becomes $0x = 0$.

$x = \frac{0}{0}$ for all $x \in \mathbb{R}$, the relation $0x = 0$ is verified.

So solution $= \mathbb{R}$; in that case we say that the equation becomes indeterminate.

Product equation

This is in the form $\mathbf{(ax + b)(cx + d) = 0}$.

Since the product of factors is null (zero) either one of them is zero. To solve this we proceed as follows:

$(ax + b)(cx + d) = 0$

$ax + b = 0$ or $cx + d = 0$

$x = -\frac{b}{a}$ or $x = -\frac{d}{c}$

Example 5.4

Solve $(2x + 4)(x - 1) = 0$.

Solution

$2x + 4 = 0$ or $x - 1 = 0$

$x = -2$ or $x = 1$.

Fractional equation of the first degree

The general form is $\frac{ax + b}{cx + d} = 0$.

We have to fix the existence condition $cx + d \neq 0$ and $\frac{ax + b}{cx + d} = 0 \Rightarrow ax + b = 0$.

We solve $ax + b$ and we take the value(s) which verify the existence conditions.

Example 5.5

Solve $\frac{2x - 6}{x + 1} = 0$.

Solution

Existence condition: $x \neq -1$

$2x - 6 = 0$

$x = 3$.

Application activity 5.1

1. Solve the following:

(a) $(x + 2)(2x - 1) = 0$ (b) $(5x - 15)(3x - 9) = 0$

(c) $x^2 - 5x = 0$ (d) $\frac{3x - 6}{x + 1} = 0$

(e) $\frac{x - 2}{5x + 3} = 0$ (f) $3(x + 7) = 0$

(e) $\frac{3 - x}{2x - 7} = 0$

Inequalities in one unknown

Activity 5.3

Discuss the difference between:

(a) linear equations and linear inequalities

(b) solving linear equations and solving linear inequalities.

The inequalities of one unknown are in the form $\mathbf{ax + b \geq 0}$ or $\mathbf{ax + b \leq 0}$ with $\mathbf{a} \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}$.

Solving inequalities

They are solved as linear equations except that:

(a) when we multiply an inequality by a negative real number the sign will be reversed

(b) when we interchange the right side and the left side, the sign will be reversed.

Example 5.6

Solve: $-2(x + 3) < 10$.

Solution

$-2x - 6 < 10$

$-2x - 6 + 6 < 10 + 6$

$-2x < 16$

$\frac{-2x}{-2} > \frac{16}{-2}$

$x > -8$

Example 5.7

Solve the following inequalities:

a) $2x + 6 \geq x - 5$

b) $\frac{6 + 4x}{12} \geq \frac{3 + 4x}{12}$

Solution

a) $2x + 6 \geq x - 5$

$2x - x + 6 \geq -5$

$x \geq -5 - 6$

$x \geq -11$

The solution set is $S = [-11, +\infty[$

b) $\frac{6 + 4x}{12} \geq \frac{3 + 4x}{12}$

$6 + 4x \geq 3 + 4x$

$4x - 4x \geq 3 - 6$

$0x \geq -3$

The solution set is $S = \mathbb{R}$.

Graphs

The graph of a linear inequality in one variable is a number line. We use an unshaded circle for < and > and a shaded circle for ≤ and ≥.

The graph for $x > -3$:

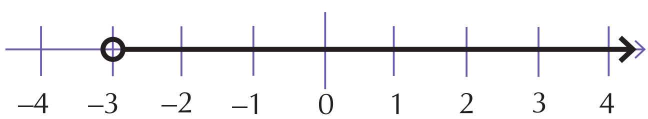


Fig 5.2

The graph for $x \geq 2$:

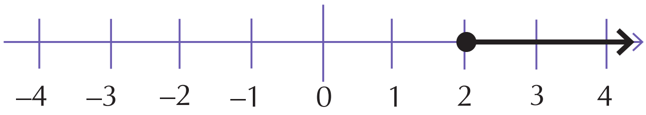


Fig 5.3

Example 5.8

Solve and graph the solution set of: $2x - 6 < 2$.

Add 6 to both sides: $2x - 6 + 6 < 2 + 6$

Divide both sides by 2: $\frac{2}{2}x < \frac{8}{z}$

$x < 4$

Open circle at 4 (since x cannot equal 4) and an arrow to the left (because we want values less than 4**).**

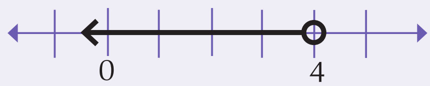


Fig 5.4

Example 5.9

Solve and graph the solution set of: $3(2x + 4) > 4x + 10$.

Multiply out the parentheses. $6x + 12 > 4x + 10$

Subtract $4x$ from both sides. $2x + 12 > 10$

Subtract 12 from both sides. $2x > -2$

Divide both sides by 2, but do not change the direction of the inequality sign, since we did not divide by a negative. $x > -1$

Unshaded circle at –1 (since x cannot equal –1) and an arrow to the right (because we want values larger than –1).

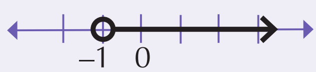


Fig 5.5

Figure 5.6 shows a graph of a linear inequality.

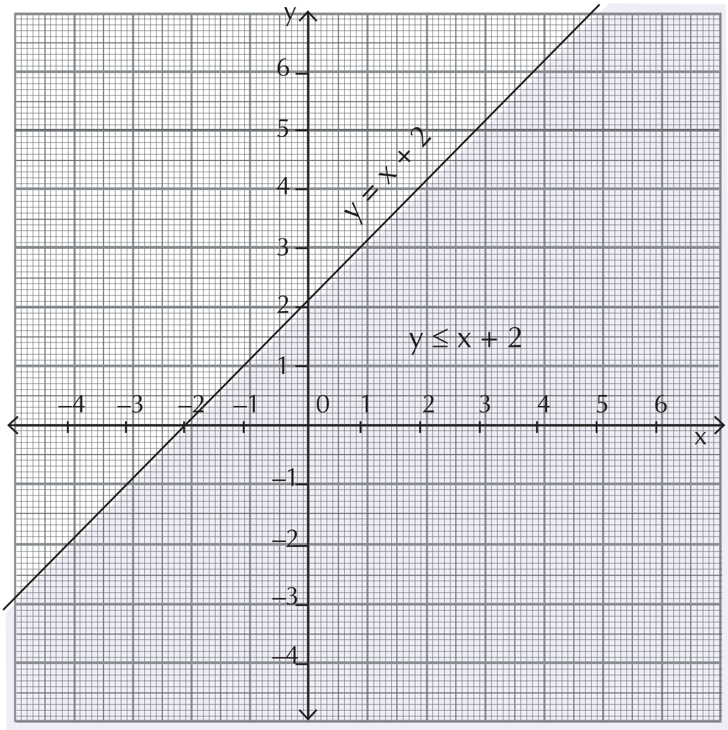


Fig 5.6

The inequality is $y \leq x + 2$.

You can see the line, $y = x + 2$ and the shaded area is where y is less than or equal to $x + 2$

Application activity 5.2

Solve and graph the following:

1. $y \leq 3$
2. $y = \frac{1}{2}x - 3$
3. $y = -3x + 2$
4. $y \geq -2$
5. $2y - x \leq 6$
6. $\frac{y}{2} + 2 > x$

5.2 Parametric equations and inequalities

Activity 5.4

Carry out research. Find the meaning of parametric equations and inequalities. Discuss your findings using suitable examples.

Parametric equations

There are also a great many curves that we cannot even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations**. Instead of defining y in terms of x i.e $y = f(x)$) or *x* in terms of *y* i.e $x = h(y)$ we define both x and y in terms of a third variable called a parameter as follows:

$$x = f(t) \qquad y = g(t)$$

This third variable is usually denoted by *t* (but does not have to be). Sometimes we will restrict the values of *t* that we shall use and at other times we will not.

If the coefficients of an equation contain one or several letters (variables) the equation is called parametric and the letters are called **real parameters**. In this case, we solve and discuss the equation (for parameters only).

Each value of t defines a point $(x, y) = (f(t),g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve**.

Example 5.10

Solve and discuss the equation $(2 - 3m)\, x + 1 = m^2(1 - x)$.

Solution

$(2 - 3m)x + 1 = m^2(1 - x)$

$2x - 3mx + 1 = m^2 - m^2x$

$2x - 3mx + m^2x - m^2 + 1 = 0$

$x(2 - 3m + m^2) = m^2 - 1$

$$x = \frac{m^2 - 1}{2 - 3m + m^2} = \frac{(m-1)(m+1)}{(m-1)(m-2)} = \frac{m+1}{m-2}$$

o solve and discuss, we have to find the set of possible values m can take (i.e. the domain) and the corresponding solutions:

$$x = \frac{m+1}{m-2}$$

The domain of $f(m) = \frac{m+1}{m-2}$ is $\mathbb{R}\setminus\{2\}$ and we can say that:

If $m \neq 2$, then the solution is $x = \frac{m+1}{m-2}$

If $m = 2$, then there is no solution.

Graphs

Sketching a parametric curve is not always an easy thing to do. Let us look at an example to see one way of sketching a parametric curve. This example will also illustrate why this method is usually not the best.

Example 5.11

Sketch the parametric curve for the following set of parametric equations.

$x = t^2 + t$ $\qquad\qquad$ $y = 2t - 1$

Solution

At this point our only option for sketching a parametric curve is to pick values of t, substitute them into the parametric equations and then plot the points.

t	x	y
–2	2	–5
–1	0	–3
$-\frac{1}{2}$	$-\frac{1}{2}$	–2
0	0	–1
1	2	1

Here is the sketch of this parametric curve.

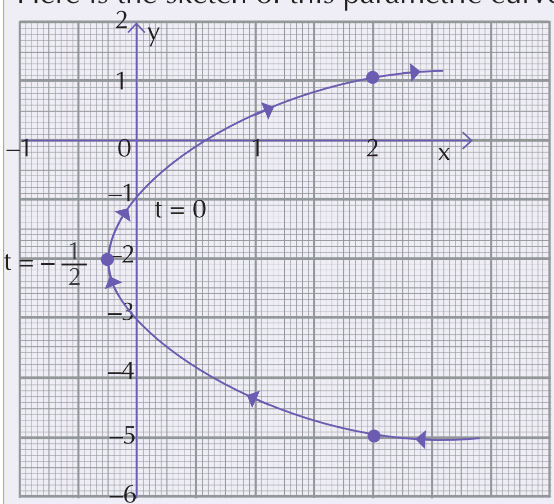


Fig 5.7

So, we have a parabola that opens to the right.

Activity 5.5

Work out the following:

1. Sketch the parametric curve for the following set of parametric equations.

   $x = t^2 + t$ $\qquad$ $y = 2t - 1$ $\qquad$ for $\qquad -1 \le t \le 1$

2. Eliminate the parameter from the following set of parametric equations.

   $x = t^2 + t$ $\qquad$ $y = 2t - 1$

3. Sketch the parametric curve for the following set of parametric equations. Clearly indicate the direction of motion.
$x = 5 \cos t$ $y = 2 \sin t$ for $0 \le t \le 2\pi$

Parametric inequalities in one unknown

Example 5.12

Solve $(m+3)x \ge 2$

Solution

$(m+3)x \ge 2$

There are 3 cases:

Case 1:

If $m = -3$ then the equation becomes

$(-3+3)x \ge 2$

$0x \ge 2$ *which is impossible*

The solution set is $S = \{\ \}$

Case 2:

If $m > -3 \Rightarrow x \ge \dfrac{2}{m+3}$, then solution set is $S = \left[\dfrac{2}{m+3}, +\infty\right[$

Case 3:

If $m < -3 \Rightarrow x \ge \dfrac{2}{m+3}$, then the solution set is $\left]-\infty, \dfrac{2}{m+3}\right]$

5.3 Simultaneous equations in two unknowns

Activity for discussion

Refer to what you learnt in previous level and answer to this question

What are simultaneous linear equations? How do we solve them?

A linear equation in two variables x and y is an equation of the form **ax + by = c** where $a \neq 0$, $b \neq 0$ and a, b, c are real numbers.

Let us consider such equation $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1, b_1, c_1, a_2, b_2 and c_2 are constants.

We say that we have two simultaneous linear equation in two unknowns or a system of two linear equation in two unknowns.

The pair (x, y) satisfying both equations is the solution of the given equation.

Example 5.13

The solution of $\begin{cases} x + y = 8 \\ x - y = 2 \end{cases}$ is (5,3)

We can solve such systems of linear equations by using one of the following methods learnt in S3: Cramer method, Comparison method and Graphical method. More other methods can be used:

1. substitution method
2. elimination method

Substitution

This method is used when one of the variables is given in terms of the other.

Example 5.14

Find the simultaneous solution of the following pair of equations: $y = 2x - 1$, $y = x + 3$.

Solution

Note that the system can also be written as $\begin{cases} y = 2x - 1 \\ y = x + 3 \end{cases}$, then

$2x - 1 = x + 3$

$x = 4$

And so $y = 4 + 3$

$y = 7$

So, the simultaneous solution is $x = 4$ and $y = 7$.

Elimination

Elimination method is used to solve simultaneous equations where neither variable is given as the subject of another.

Example 5.15

Solve simultaneously, by elimination: $\begin{cases} 5x + 3y = 12 \\ 7x + 2y = 19 \end{cases}$

Solution

$\begin{cases} 5x + 3y = 12 \ldots \ldots (1) \\ 7x + 2y = 19 \ldots \ldots (2) \end{cases}$

We multiply (1) by 2 and (2) by –3:

$$\begin{cases} 10x + 6y = 24 \\ -21x - 6y = -57 \end{cases}$$

Adding the two equations term by term gives:

$-11x = -33$

$x = 3$

Substituting $x = 3$ into (1) gives:

$5(3) + 3y = 12$

$15 + 3y = 12$

$3y = -3$

$y = -1$

Hence $x = 3$, $y = -1$ is the solution to the system of equations.

5.4 Applications

Systems of two equations have a wide practical application whenever decisions arise. Decisions of this nature always involve two unknown quantities or variables. The following steps are hereby recommended in order to apply simultaneous equations for practical purposes.

1. Define variables for the two unknown quantities, in case they are not given.
2. Formulate equations using the unknown variables and the corresponding data.
3. Solve the equations simultaneously.

Example 5.16

At a clearance sale all, CDs are sold for one price and all DVDs are sold for another price. Mugabo bought 3 CDs and 2 DVDs for a total of 1,900 FRW, and Mucyo bought 2 CDs and 5 DVDs for a total of 3,100 FRW. Find the cost of each item.

Solution

Let x be the cost of one CD and y be the cost of one DVD. Then

3 CDs and 2 DVDs cost 1,900 FRW, so $3x + 2y = 1{,}900$ ...(1)

2 CDs and 5 DVDs cost 3,100 FRW, so $2x + 5y = 3{,}100$ ...(2)

We have to solve the following system of equation

$$\begin{cases} 3x + 2y = 1{,}900 \\ 2x + 5y = 3{,}100 \end{cases}$$

We will eliminate x by multiplying equation (1) by 2 and equation (2) by –3.

$$\begin{cases} 6x + 4y = 3{,}800 \\ -6x - 15y = -9{,}300 \end{cases}$$

$-11y = -5,500$

$y = 500$

Substituting in equation (1) gives $3x + 2(500) = 1,900$

$3x + 1,000 = 1,900$

$3x = 900$

$x = 300$

So, the cost of one CD is 300 FRW and the cost of one DVD is 500 FRW.

Application activity 5.3

Form equations to solve the following.

1. Kigali University canteen sells fish fillet and egg rolls to university students. The cost of fish fillet is x FRW per piece, while egg rolls costs y FRW per piece. A fish fillet costs 500 FRW more than an egg roll. If the total cost of a fish fillet and an egg roll is 2,000 FRW, what is the unit price of:

   (a) a fish fillet (b) an egg roll?

2. The Coca-Cola bottling company distributes x crates of Fanta and y crates of Coke within Kigali area every day. The daily distribution of Fanta is 100 crates less than that of Coke. If the company distributes 400 crates daily, how many crates can be distributed for each brand?

3. Rutaya has joined University of Rwanda. He is given 10,000 FRW for pocket money. He decides to spend x FRW on box files and y FRW on file folders. If Rutaya buys 3 box files and 2 file folders he uses all the money. If he buys 1 box file and 2 file folders, he spends 5,000 FRW. Determine the unit price of:

   (a) a box file (b) a file folder.

4. A construction company transports material to different sites. The cost of transporting sand to site 1 is 6,500 FRW per trip; while gravel is transported to the same site at 10,000 FRW per trip. The cost of transporting sand to site 2 is 10,000 FRW per trip while gravel is transported at 13,000 FRW per trip. Assuming the total transport costs to sites 1 and 2 were 63,000 FRW and 90,000 FRW respectively, how many trips did the vehicle make to each site?

5. The publisher of Ingabo Magazine wants to print a new book. The book can either be hard cover or paperback. It takes 3 minutes to bind a hard cover book and 2 minutes to bind a paperback. The total time available for binding is 500 hours. If the total number of hard cover editions and paperbacks is 12,000, determine the total number of hard cover books and paperbacks that can be printed.

Activity 5.6

Linear equations and inequalities have their use in our everyday life. Research on these and present your findings for discussion in class.

Distance = rate x time

In this equation, for any given steady rate, the relationship between distance and time will be linear. However, distance is usually expressed as a positive number, so most graphs of this relationship will only show points in the first quadrant. Notice that the direction of the line in the graph of Figure 5.8 is from bottom left to top right. Lines that tend in this direction have positive slope. A positive slope indicates that the values on both axes are increasing from left to right.

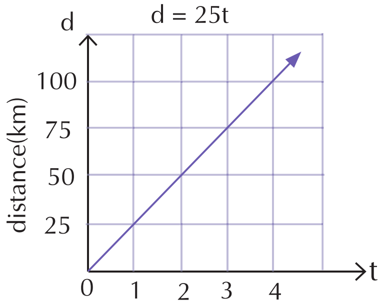


Fig 5.8 Graph of the relationship between distance and time when rate is a constant 25 km per hour.

Amount of water in a leaking bucket = rate of leak x time

In this equation, since you cannot have a negative amount of water in the bucket, the graph will also show points only in the first quadrant. Notice that the direction of the line in this graph is top left to bottom right. Lines that tend in this direction have negative slope. A negative slope indicates that the values on the y axis are decreasing as the values on the x axis are increasing.

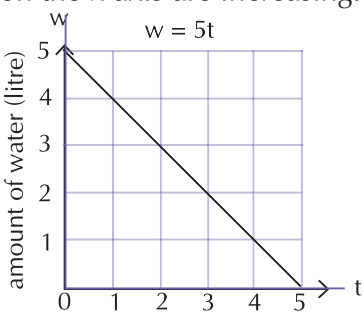


Fig. 5.9 Graph of the relationship between amount of water and time when rate of leak is a constant 1 litre per minute.

Number of angles of a polygon = number of sides of that polygon

In this graph, we are relating values that only make sense if they are positive, so we show points only in the first quadrant. In this case, since no polygon has fewer than 3 sides or angles, and since the number of sides or angles of a polygon must be a whole number, we show the graph starting at (3,3) and indicate with a dashed line that points between those plotted are not relevant to the problem.

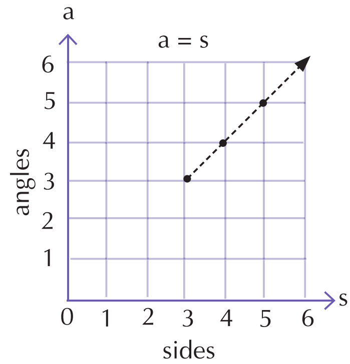


Fig 5.10 Graph of the relationship between number of angles and number of sides of a polygon.

Degrees Celsius = $\frac{5}{9}$ (degrees Fahrenheit – 32)

Since it is perfectly reasonable to have both positive and negative temperatures, we plot the points on this graph on the full coordinate grid.

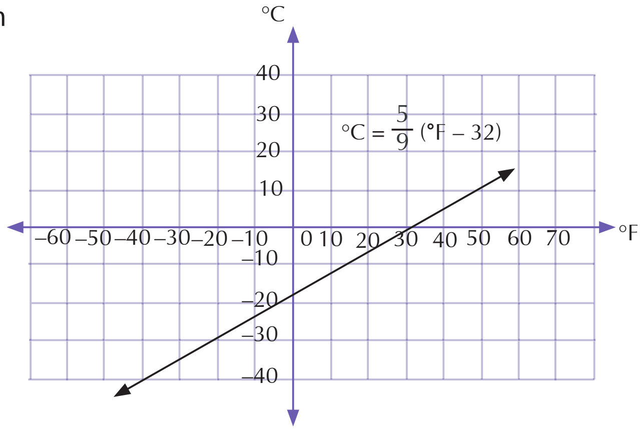


Fig 5.11 Graph of the relationship between degrees Celsius and degrees Fahrenheit.

Application activity 5.4

1. The price of a machine has been increased by 15% and is being sold for 78,500 FRW.
   How much did the store pay the manufacturer of the machine?
2. Two cars are 500 kilometres apart and moving directly towards each other. One car is moving at a speed of 100 kph and the other is moving at 70 kph. Assuming that the cars start moving at the same time, how long does it take for the two cars to meet?

3. An electronics shop sells DVD players and mobile phones. The shop makes a 7,500 FRW profit on the sale of each DVD player (d) and a 3,000 FRW profit on the sale of each mobile phone (c). The store wants to make a profit of at least 25,500 FRW from its sales of DVD players and mobile phones. Which inequality correctly describes this situation?

   a) $7{,}500d + 3{,}000c < 25{,}500$
   b) $7{,}500d + 3{,}000c \leq 25{,}500$
   c) $7{,}500d + 3{,}000c > 25{,}500$
   d) $7{,}500d + 3{,}000c \geq 25{,}500$

Summary

1. The general form of a linear equation is $\mathbf{ax + b = 0}$.
2. The inequalities of one unknown are of the form $ax + b \geq 0$ or $ax + b \leq 0$ with $a \in \mathbb{R}$, $b \in \mathbb{R}$.
3. A **product equation** is one of the form $(ax + b)(cx + d) = 0$
4. The general form of a **fractional** equation of the first degree is: $\mathbf{\frac{ax + b}{cx + d} = 0}$
5. In the case where certain coefficients of an equation contain one or several variables, the equation is called **parametric** and the letters are called real **parameters**.
6. A system of two linear equations in two unknowns is of the form $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1, b_1, c_1, a_2, b_2 and c_2 are constants.
7. We can solve systems of linear equations by using one of the following methods:

   a. substitution method
   b. elimination method.

Topic area: Algebra

Sub-topic area: Equations and inequalities

Unit 6

Quadratic equations and inequalities

Key unit competence

Model and solve algebraically or graphically daily life problems using quadratic equations or inequalities.

6.0 Introductory activity

Smoke jumpers are fire fighters who parachute into areas near forest fires. Jumpers are in free fall from the time they jump from a plane until they open their parachutes. The function $y = -16t^2 + 1600$ gives a jumper's height y in metre after t seconds for a jump from $1600m$.

a) What is the highest exponent of the unknown in the function

y = f(t) $y = -16t^2 + 1600$? is this equation linear?

b) Complete a table of values for t = 0, 1, 2, 3, 4, 5 and 6 and try to plot the graph of y = f(t) $y = -16t^2 + 1600$

6.1 Introduction

In Senior 3, we learnt about quadratic equations and ways to solve them.

Activity 6.1

Carry out research to obtain the definition of quadratic equation. Discuss your findings with the rest of the class.

The term **quadratic** comes from the word *quad* meaning square, because the variable gets squared (like $\mathbf{x^2}$). It is also called an "equation of degree 2" because of the "2" on the **x**.

The **standard form** of a quadratic equation looks like this:

$\mathbf{ax^2 + bx + c = 0}$

where **a**, **b** and **c** are known values and **a** cannot be 0.

"**x**" is the **variable** or the unknown.

Here are some more examples of quadratic equations:

$2x^2 + 5x + 3 = 0$ In this one, **a = 2**, **b = 5** and **c = 3**

$x^2 - 3x = 0$ For this, **a = 1**, **b = –3** and **c = 0**, so 1 is not shown.

$5x - 3 = 0$ This one is **not** a quadratic equation. It is missing a value in **x^2** i.e **a = 0**, which means it cannot be quadratic.

6.2 Equations in one unknown

A quadratic equation in the unknown x is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are given real numbers, with $a \neq 0$. This may be solved by **completing the square** or by using the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If b2 – 4ac > 0, there are two distinct real roots
- If b2 – 4ac = 0, there is a single real root (which may be convenient to treat as two equal or coincident roots)
- If b2 – 4ac < 0, the equation has no real roots.

We know that the quadratic equation is of the form:

$ax^2 + bx + c = 0$ where $a \neq 0$. Let us get a formula to solve x.

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Let $\Delta = b^2 - 4ac$, we get $\left(x + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{\Delta}{4a^2}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{\Delta}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Remember that to solve for x:

$$x = \frac{-b \pm (b^2 - 4ac)}{2a}$$

Example 6.1

Solve

$x^2 - 3x + 2 = 0$

Solution

$x^2 - 3x + 2 = 0$; $a = 1$, $b = -3$, $c = 2$

$\Delta = b^2 - 4ac = (-3)^2 - 4(1)(2) = 9 - 8 = 1$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{3 \pm 1}{2}$$

$x = 2$ or $x = 1$

Application activity 6.1

Solve the following quadratic equations:

1. $16x^2 - 8x + 1 = 0$
2. $x^2 - \frac{x}{3} + \frac{2}{7} = 0$
3. $x^2 - 6x + 5 = 0$
4. $2x^2 - 3x - 2 = 0$
5. $2x^2 - 3x - 1 = 0$
6. $2x^2 + 5x + 4 = 0$
7. $x^2 + 4x + 4 = 0$

Sum and product of roots

Consider $ax^2 + bx + c = 0$ (where $a \neq 0$)

If $\Delta > 0$, $x_1 = \frac{-b + \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b - \sqrt{\Delta}}{2a}$

The sum (S) is $x_1 + x_2 = \frac{-b + \sqrt{\Delta}}{2a} + \frac{-b - \sqrt{\Delta}}{2a} = \frac{-b + \sqrt{\Delta} -b + \sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$

The product (P) is

$$x_1 \times x_2 = \frac{-b+\sqrt{\Delta}}{2a} \times \frac{-b-\sqrt{\Delta}}{2a} = \frac{b^2-\Delta}{4a^2} = \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

So we can conclude that two numbers whose sum is S and product is P are the roots of the quadratic equations $x^2 - Sx + P = 0$.

Example 6.2

Find two numbers which have the sum 17 and the product 30.

Solution

Let the two numbers be x_1 and x_2, then

$$\begin{cases} \text{Sum } (S) = x_1 + x_2 = 17 \\ \text{Product } (P) = x_1 \times x_2 = 30 \end{cases}$$

$x^2 - Sx + P = 0$ $\qquad$ $x^2 - 17x + 30 = 0$

And we can also solve

$x^2 - 17x + 30 = 0$

$\Delta = 289 - 120 = 169$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{17 \pm \sqrt{169}}{2(1)} = \frac{17 \pm 13}{2}$$

$$x = \frac{30}{2} = 15 \text{ or } x = \frac{4}{2} = 2$$

The two numbers are 2 and 15.

Application activity 6.2

Solve the following equations

1. $x^2 + 10x - 7 = 0$
2. $15 - x^2 - 2x = 0$
3. $x^2 - 3x = 4$
4. $12 - 7x + x^2 = 0$
5. $2x - 1 + 3x^2 = 0$
6. $x(x + 7) + 6 = 0$
7. $2x^2 - 4x = 0$
8. $x(4x + 5) = -1$
9. $2 - x = 3x^2$
10. $6x^2 + 3x = 0$
11. $x^2 + 6x = 0$
12. $x^2 = 10x$
13. $x(4x + 1) = 3x$
14. $20 + x(1 + x) = 0$

15. $x(3x - 2) = 8$

16. $x^2 - x(2x - 1) = 2x$

17. $x(x - 1) = 2x$

18. $4 + x^2 = 2(x + 2)$

19. $x(x - 2) = 3$

20. $1 - x^2 = x(1 - x)$

Activity 6.2

In pairs, form the quadratic equations that have 7 and –3 as roots.

Solving quadratic equations by factorizing

Let us use an example, $x^2 - 5x + 6$. To solve $x^2 - 5x + 6 = 0$ we must first factorise $x^2 - 5x + 6$. To do this we have to find two numbers with a sum of –5 and a product of 6. The numbers required are –2 and –3, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

In solving this, we use the following law:

When the product of two or more numbers is zero, then at least one of them must be zero. So if $ab = 0$ then $a = 0$ or $b = 0$.

Example 6.3

Solve for x:

$x^2 - 3x + 2 = 0$

Solution

$x^2 - 3x + 2 = 0$

We need two numbers with sum –3 and product 2. These are –1 and –2.

$x^2 - 3x + 2 = (x - 1)(x - 2) = 0$

$x - 1 = 0$ or $x - 2 = 0$

$x = 1$ or 2

Application activity 6.3

Solve for x:

a) $x^2 + 7x + 10 = 0$

b) $x^2 + 6x + 8 = 0$

c) $x^2 + 11x + 10 = 0$

d) $x^2 - 8x + 12 = 0$

e) $x^2 - 5x + 4 = 0$

f) $x^2 - 11x + 24 = 0$

g) $3x^2 + 21x + 30 = 0$

h) $2x^2 + 4x - 30 = 0$

i) $2x^2 - 24x + 72 = 0$

j) $3x^2 - 21x + 36 = 0$

k) $5x^2 - 5x - 210 = 0$ l) $4x^2 + 32x + 48 = 0$
m) $x^2 - 14x = 15$ n) $x^2 + 2 = 3x$
o) $x^2 = 3x + 28$ p) $x^2 = 20 + x$
q) $x(x + 2) = 15$ r) $x(x - 2) = 5(x + 12)$

6.3 Inequalities in one unknown

The product ab of two factors is positive if and only if
(i) $a > 0$ and $b > 0$ or (ii) $a < 0$ and $b < 0$.

Thus $(x - 1)(x + 2) > 0$ if and only if

(i) $x - 1 > 0$ and $x + 2 > 0$ or (ii) $x - 1 < 0$ and $x + 2 < 0$

i.e., $x > 1$ and $x > -2$ or $x < 1$ and $x < -2$

Sign diagrams

Although this method is sound it is not of much practical use in more complicated problems. A better method which is useful in more complicated problems is the following which uses the ''sign diagram'' of the product $(x - 1)(x + 2)$.

$(x - 1)(x + 2) > 0$

The critical values are $x = 1$ and $x = -2$. (i.e., the values of x at which the factor is zero.)

The sign diagram of $(x - 1)(x + 2)$ is thus:

Factors \ x	$-\infty$		-2		1		$+\infty$
$x - 1$		$-$		$-$	0	$+$	
$x + 2$		$-$	0	$+$		$+$	
$(x - 1)(x + 2)$		$+$	0	$-$	0	$+$	

Fig. 6.1

The answer is $x < -2$ or $x > 1 \Leftrightarrow x \in]-\infty, -2[\cup]1, +\infty[$

Note: The sign diagram of a linear function $ax + b$ is summarized here.

The critical value is $x = -\frac{b}{a}$ (it is the value of x when $ax + b = 0$).

Factors \ x	$-\infty$		$-\frac{b}{a}$		$+\infty$
$ax + b$		Opposite sign of a	0	Same sign as a	

Fig. 6.2

For any interval, the sign of the product of two linear functions is the product of the signs of its factors.

Example 6.4

Solve the inequality $(x + 3)(x - 2) > 0$.

Solution

The critical values are $x = -3$, $x = 2$.

The required sign diagram is:

x / Factors	$-\infty$		-3		2		$+\infty$
$x + 3$		$-$	0	$+$		$+$	
$x - 2$		$-$		$-$	0	$+$	
$(x + 3)(x - 2)$		$+$	0	$-$	0	$+$	

Fig. 6.3

The answer is $\mathbf{x} < -3$ **or** $\mathbf{x} > 2 \Leftrightarrow x \in]-\infty, -3[\cup]2, +\infty[$

Note: The summary of sign diagram for $ax^2 + bx + c$:

If $\Delta > 0$, there are two critical values, x_1 and x_2

x / Factors	$-\infty$		x_1		x_2		$+\infty$
$ax^2 + bx + c$	Same sign as a		0	Opposite sign of a	0	Same sign as a	

Fig. 6.4

If $\Delta = 0$, there is one critical value x_1 and the quadratic has the same sign as a.

If $\Delta < 0$, there is no critical value and the quadratic has the same sign as a.

Application activity 6.4

Solve the following inequalities

1. $(1 - x)(2x + 1) > 0$
2. $2x^2 - x - 3 < 0$
3. $x(x + 4) > x - 4$
4. $x(x + 4) > x - 1$
5. $x^2 + x + 1 < 0$

Inequalities depending on the quotient of two linear factors

The quotient $\frac{a}{b}$ of two factors is positive if and only if

(i) $a > 0$ and $b > 0$ or (ii) $a < 0$ and $b < 0$.

These are the same conditions under which the product ab is positive. Therefore solving inequalities such as $\frac{x+3}{x-1} > 0$ is the same as solving $(x + 3)(x - 1) > 0$.

Example 6.5

Solve the inequality $\frac{2x-1}{x+2} < 0$

Solution

The critical values are $x = \frac{1}{2}$, $x = -2$.

The sign diagram is:

x / Factors	$-\infty$		-2		$\frac{1}{2}$		$+\infty$
$2x - 1$		$-$	0	$-$	0	$+$	
$x + 2$		$-$		$+$		$+$	
$\frac{2x-1}{x+2}$		$+$	0	$-$	0	$+$	

Fig. 6.5

The answer is $-2 < x < \frac{1}{2} \Leftrightarrow x \in]-2, \frac{1}{2}[$

Application activity 6.5

Solve the following inequalities

1. $\frac{2x-1}{x+2} > 4$
2. $\frac{2x-1}{x+2} > 2$
3. $\frac{3x+2}{2-x} \geq 4$

Solving general inequalities

The techniques illustrated in the previous pages can be used to solve complicated inequalities. There are also other techniques which may be used in special cases.

Example 6.6

Solve the inequality $\frac{(x-1)(x^2+1)}{x+2} > 0$

Solution

The critical values are $x = 1$, $x = -2$. (Note: $x^2 + 1$ is positive definite and does not contribute any critical value.)

The sign diagram is:

Factors \ x	$-\infty$		-2		1		$+\infty$
$x-1$		$-$		$-$	0	$+$	
$x+2$		$-$	0	$+$		$+$	
x^2+1		$+$		$+$		$+$	
$\frac{(x-1)(x^2+1)}{x+2}$		$+$		$-$	0	$+$	

Fig. 6.6

The answer is $\mathbf{x < -2}$ **or** $\mathbf{x > 1} \Leftrightarrow \mathbf{x} \in]-\infty, \mathbf{-2}[\cup]\mathbf{1}, +\infty[$

Application activity 6.6

Solve the following inequalities:

1. $\frac{(x-1)(x+1)^2}{x+2} \geq 0$
2. $|x+2| > |2x-1|$
3. $2x - 4 < x + 7$
4. $\frac{2}{x+1} > 2$
5. $\frac{x+1}{x-1} < 5$
6. $\frac{1-2x}{3} \leq \frac{x-1}{2}$
7. $\frac{(x-2)(x-1)}{(x+1)(x-3)} > 0$
8. $\frac{x}{x-1} < \frac{x}{a-2}$
9. $\frac{x+2}{(x+1)(x-3)} < 0$
10. $2 - x > 2x + 4 > x$
11. $x - 1 < 3x + 1 < x + 5$
12. $(x+1)(x+3)(x+5) > 0$

6.4 Parametric equations

In case certain coefficients of equations contain one or several letter variables, the equation is called **parametric** and the letters are called **real parameters**. In this case, we solve and discuss the equation (for parameters only).

Example 6.7

Solve and discuss the equation $(2 - 3m)x + 1 = m^2(1 - x)$

Solution

$(2 - 3m)x + 1 = m^2(1 - x)$

$2x - 3mx + 1 = m^2 - m^2x$

$2x - 3mx + m^2x - m^2 + 1 = 0$

$x(2 - 3m + m^2) - m^2 + 1 = 0$

$x(2 - 3m + m^2) = m^2 - 1$

$$x = \frac{m^2 - 1}{2 - 3m + m^2} = \frac{(m-1)(m+1)}{(m-1)(m-2)} = \frac{m+1}{m-2}$$

If $m = 2$, then there is no solution.

If $m \neq 2$, then the solution is $x = \frac{m+1}{m-2}$

Parametric equations in one unknown

If at least one of the coefficients a, b and c depends on the real parameter which is not determined, the root of the parametric quadratic equation depends on the values attributed to that parameter.

Example 6.8

Find the values of k for which the equation $x^2 + (k + 1)x + 1 = 0$ has:

(a) two distinct real roots

(b) no real roots.

Solution

$\Delta = (k + 1)^2 - 4(1)(1) = (k + 1)^2 - 4$

$= k^2 + 2k + 1 - 4 = k^2 + 2k - 3 = (k + 3)(k - 1) = 0$ then $k = -3$ or $k = 1$.

Table of sign of $\Delta = k^2 + 2k - 3 = (k + 3)(k - 1)$

k / Factors	$-\infty$		-3		1		$+\infty$
$k + 3$		$-$	0	$+$		$+$	
$k - 1$		$-$		$-$	0	$+$	
$(k + 3)(k - 1)$		$+$	0	$-$	0	$+$	

Fig. 6.7

a) For two distinct real roots; $\Delta > 0$ and so $k < -3$ or $k > 1$.

b) For no real roots $\Delta < 0$ and so $-3 < k < 1$.

Application activity 6.7

1. Find the range of values of m for which the equation $(m - 3)x^2 - 8x + 4 = 0$ has
   (a) two real roots (b) no real root
   (c) one double root.
2. Find the range of values of k for which the equation $x^2 - 2(k + 1)x + k^2 = 0$ has:
   (a) two real roots (b) no real root
   (c) one double root.
3. Find the range of values of m for which the equation $2x^2 - 5x + 3m - 1 = 0$ has
   (a) two real roots (b) no real root
   (c) one double root.
4. Find the set of values of m for which $x^2 + 3mx + m$ is positive for all real values of x.

6.5 Simultaneous equations in two unknowns

To solve simultaneous equations involving a quadratic equation we use substitution of one equation into the other.

Example 6.9

Solve simultaneously: $y = x^2$ and $y = 2x + 3$.

Solution

The system can be written as

$$\begin{cases} y = x^2 \\ y = 2x + 3 \end{cases}$$

We substitute x^2 for y in the second equation.

So, $x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x - 3 = 0$ or $x + 1 = 0$

$x = 3$ or $x = -1$

When $x = 3$, $y = 3^2 = 9$

When $x = -1$, $y = (-1)^2 = 1$

Thus the solutions are $x = 3, y = 9$ or $x = -1, y = 1$.

Application activity 6.8

1. Solve the following simultaneously:

   (a) $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

   (b) $\begin{cases} y = x^2 \\ y = x + 6 \end{cases}$

   (c) $\begin{cases} y = x^2 \\ y = x + 12 \end{cases}$

   (d) $\begin{cases} y = x^2 \\ y = 2x - 1 \end{cases}$

   (e) $y = x^2$ and $y - 14x = -48$

   (f) $y = x^2$ and $x + y = 12$

   (g) $y = 8 - x$ and $xy = 7$

   (h) $xy = -5$ and $y = x - 6$

   (i) $xy = 4$ and $y = 4 - x$

   (j) $xy + 6 = 0$ and $x + y = 1$

2. Two numbers are such that one number is the square of the other number and is also 8 more than twice that number. What are the two numbers?

6.6 Applications of quadratic equations and inequalities

Activity 6.3

In groups of five, research on the importance and necessity of quadratic equations and inequalities.

Quadratic equations lend themselves to modelling situations that happen in real life. These include:

- Projectile motions
- The rise and fall of profits from selling goods
- The decrease and increase in the amount of time it takes to run a kilometre based on your age, and so on.

The wonderful part of having something that can be modelled by a quadratic is that you can easily solve the equation when set equal to zero and predict. If you throw a ball (or shoot an arrow, fire a missile or throw a stone) it will go up into the air, slowing down as it goes, then come down again. A quadratic equation can tell you where it will be at any given time.

Example 6.10

A ball is thrown straight up, from 3 m above the ground, with a velocity of 14 m/s. After how long does it hit the ground?

Solution

Ignoring air resistance, we can work out its height by adding up these three things:

The height starts at 3 m:	3
It travels upwards at 14 metres per second (14 m/s):	14t
Gravity pulls it down, changing its speed by *about* 5 m/s per second (5 m/s²):	$-5t^2$
(Note: the $-5t^2$ is simplified from $-(\quad)at^2$ with a = 9.81 m/s²)	

Add them up and the height **h** at any time **t** is:

$h = 3 + 14t - 5t^2$

And the ball will hit the ground when the height is zero:

$3 + 14t - 5t^2 = 0$

We can rewrite it as

$-5t^2 + 14t + 3 = 0$

Let us solve it; we can use factoring method.

Multiply all terms by −1 to make it easier: $5t^2 - 14t - 3 = 0$

Now we can factor it.

a x c = **−15**, and b = **−14**.

The positive factors of −15 are 1, 3, 5, 15, and one of the factors has to be negative.

By trying a few combinations we find that −15 and 1 work (−15x1 = −15, and −15+1 = −14)

Rewrite middle with – 15 and 1: $5t^2 - 15t + t - 3 = 0$

Factor first two and last two: $5t(t - 3) + 1(t - 3) = 0$

Common factor is (t – 3): $(5t + 1)(t - 3) = 0$

And the two solutions are: $5t + 1 = 0$ or $t - 3 = 0$

The "t = – 0.2" is a negative time, impossible in our case.

The "t = 3" is the answer we want:

The ball hits the ground after 3 seconds.

Example 6.11

A company is going to make frames as part of a new product they are launching. The frames will be cut out of a piece of steel, and to keep the weight down, the final area should be **28 cm².** The inside of the frame has to be **11 cm by 6 cm.** What should the width **x** of the metal be?

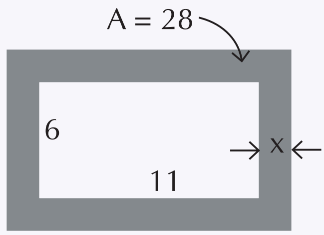


Fig. 6.8

Area of steel before cutting:

Area = (11 + 2x) x (6 + 2x) cm^2

Area = 66 + 22x + 12x + $4x^2$

Area = $4x^2$ + 34x + 66

Area of steel after cutting out the 11 x 6 middle:

Area = $4x^2$ + 34x + 66 - 66

Area = $4x^2$ + 34x

Now, solving graphically:

Here is the graph of $4x^2 + 34x$:

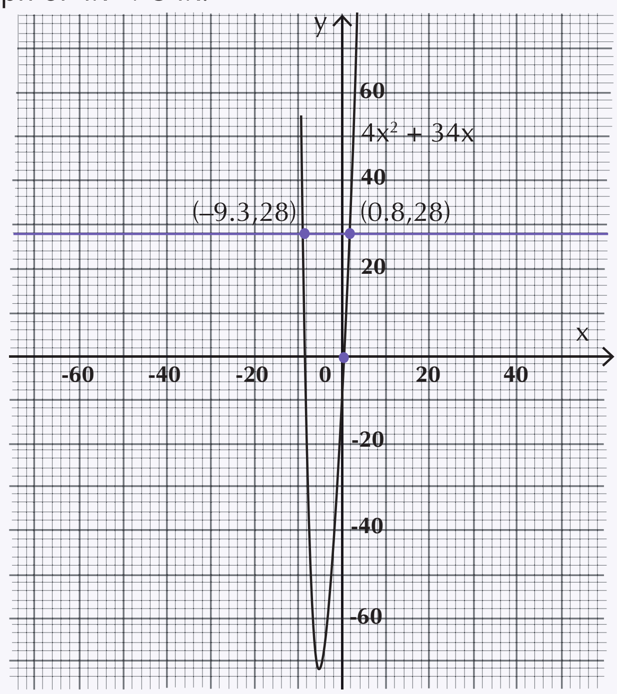


Fig. 6.9

The desired area of **28** is shown as a horizontal line.
The area equals 28 cm^2 when:

x is about –9.3 or 0.8

The negative value of **x** makes no sense, so the answer is:

x = 0.8 cm (approx.)

Application activity 6.9

1. The height of a ball *t* seconds after it is thrown into the air from the top of a building can be modelled by $h(t) = -16t^2 + 48t + 64$, where $h(t)$ is height in metres. How high is the building? How high does the ball rise before starting to drop downward, and after how many seconds does the ball hit the ground?
2. The profit function telling George how much money he will net for producing and selling *x* specialty umbrellas is given by $P(x) = -0.00405x^2 + 8.15x - 100$.

   What is George's loss if he doesn't sell any of the umbrellas he produces? How many umbrellas does he have to sell to break even, and how many does he have to sell to earn the greatest possible profit?
3. A ball is thrown upwards with a velocity of 40 m/s from a height of 6 m. At what times is the height of the ball greater than 12 m? Use the height formula $h = -16t^2 + v_0t + h_0$ and round your answers to the nearest hundredths of a second.

Summary

1. A **quadratic equation** in the unknown x is an equation of the form

   $\mathbf{ax^2 + bx + c = 0}$, where a, b and c are given real numbers, with $a \neq 0$.
2. The product of two factors is positive if and only if

   (i) $a > 0$ *and* $b > 0$ or (ii) $a < 0$ *and* $b < 0$.
3. If at least one of the coefficients a, b and c depend on the **real parameter** which is not determined, the root of the **parametric quadratic equation** depends on the values attributed to that parameter.

Topic area: Analysis

Sub-topic area: Functions

Unit 7

Polynomial, rational and irrational functions

Key unit competence

Use concepts and definitions of functions to determine the domain of rational functions and represent them graphically in simple cases and solve related problems.

7.0 Introductory activity

1. Consider the following sentences:
   i. The function of the heart is to pump blood
   ii. Last Saturday, my sister got married; the arrangement of chairs in the main hall was in function of the number of guests.
   iii. The area of a square is function of the length of its side.

Explain what is meant by the word "Function" in each of the three sentences above.

2. Any function involves at least two variables. Identify the "independent variable" and the "dependent variable" in the following functions:
   i. $y=\frac{4x-4}{(x-1)^2}$
   ii. $A=\pi r^2$
3. Classify the following functions as "polynomial", "rational" or "irrational"

   a) $f(x)=\sqrt{\frac{x^2+1}{x-2}}$ b) $f(x)=\frac{x+1}{x-5}$ c) $f(x)=\sqrt{x^2-1}$

   d) $f(x)=2x-7$ e) $f(x)=\frac{x^3+2x-4}{5x}$

7.1 Polynomials

Activity 7.1

Research on the definition of polynomial. Discuss with the rest of the class to come up with a concise definition.

Polynomial comes from *poly-* meaning "many" and *-nomial* meaning "term".

A polynomial looks like this:

$$4xy^2 + 3x - 5$$

terms

example of a polynomial

this one has 3 terms

A polynomial can have:

constants (like **3**, **–20**, or   )

variables (like **x** and **y**)

exponents (like the 2 in y^2), but only **0, 1, 2, 3, ...** etc

that can be combined using **addition, subtraction, multiplication and division.**

A polynomial can never be divided by a variable such as $\frac{3}{x}$.

Generally

A function f given by $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$, where $a_0, a_1, .., a_n$ are constants with $a_n \neq 0$, is called a polynomial function of degree n.

These **are** polynomials:

- **7x**
- **y – 2**
- $\mathbf{512v^5 + 99w^5}$
- **4**
- $\frac{x}{2}$ is a polynomial, because you can divide by a constant

And these are **not** polynomials:

- $\mathbf{3xy^{-2}}$ is not, because the exponent is "–2" (exponents can only be 0, 1, 2,...)
- $\frac{2}{(x+2)}$ is not, because dividing by a variable is not allowed
- $\sqrt{x}$ is not, because the exponent is "   "
- $\sqrt{2}$ is allowed, because it is a constant (= 1.4142...)

Application activity 7.1

Which of the following are polynomials? Which ones are not? Give reasons for your responses.

1. 9
2. $-6x^2 - (\frac{7}{9})z$
3. $3xyz + 3xy^2z - 0.1xz - 200y + 0.5$

There are special names for polynomials with 1, 2 or 3 terms:

$3xy^2$	$5x - 1$	$3x + 5y^2 - 3$
Mononial (1 term)	Binomial (2 terms)	Trinomial (3 terms)

Polynomials can have no variable at all:

For example, 23 is a polynomial. It has just one term, which is a constant.

Or one variable:

For example, $x^4 - 2x^2 + x$ has three terms, but only one variable (x)

Or two or more variables:

For example: $xy^4 - 5x^2z$ has two terms, and three variables (x, y and z)

The **degree** of a polynomial with only one variable is the **largest exponent** of that variable. For example, in

$4x^3 - x + 3$ the degree is **3** the largest exponent of *x*.

Factorization of polynomials

Activity 7.2

In groups, discuss the meaning of factorization. What are the different operations involved in factorization?

In arithmetic, you are familiar with factorization of integers into prime factors.

For example, 6 = 2 x 3.

The 6 is called the **multiple**, while 2 and 3 are called its **divisors or factors**.

The process of writing 6 as product of 2 and 3 is called **factorization.** Factors 2 and 3 cannot be further reduced into other factors.

Like factorization of integers in arithmetic, we have factorization of polynomials into other irreducible polynomials in algebra.

For example, $x^2 + 2x$ is a polynomial. It can be factorized into x and (x + 2). $x^2 + 2x = x(x + 2)$.

So, x and $x + 2$ are two factors of $x^2 + 2x$. While x is a monomial factor, $x + 2$ is a binomial factor.

Factorization is a way of writing a polynomial as a product of two or more factors. Here is an example:

$x^3 - 3x^2 - 2x + 6 = (x - 3)(x^2 - 2) = (x - 3)(x + \sqrt{2})(x - \sqrt{2})$

Types of factorization

1. Factorization into monomials

Remember the distributive property? It is $a(b + c) = ab + ac$

Example 7.1

Factorize $ax + bx$

Solution:

In ax and bx, x is common and also a common factor.
Write the common factor x in the polynomial $ax + bx$ outside as $x(a + b)$
Now, both x and $a + b$ are factors of the polynomial
$ax + bx$
So, factorization of
$ax + bx = x(a + b)$
Here the common factor x is a monomial.

2. By grouping of terms

Example 7.2

Factorize

$a^2 + bc + ab + ac$

Solution:

Let us group terms in $a^2 + bc + ab + ac$
But, which terms to group?
Those terms which yield a common factor on grouping. Group the terms like this:
$a^2 + ab + bc + ac$
a is the common factor in $a^2 + ab$ and c is the common factor in $bc + ac$
$a^2 + ab = a(a + b)$ and $bc + ac = c(b + a) = c(a + b)$
Now, the common factor is the binomial $(a + b)$.

So, factorization of

$a^2 + ab + bc + ac =$

$a(a + b) + c(a + b) =$

$(a + b)(a + c)$

$(a + b)$ and $(a + c)$ are two binomial factors into which $a^2 + ab + bc + ac$ is factorized.

3. Factorization of trinomial perfect squares

Trinomial perfect squares have three monomials, in which two terms are perfect squares and one term is the product of the square roots of the two terms which are perfect squares

Example 7.3

$a^2 + 2ab + b^2$

In this trinomial, a^2 and b^2 are the two perfect squares and $2ab$ is the product of the square roots of a^2 and b^2.

Factorization of $a^2 + 2ab + b^2$ gives the famous formula:

$(a + b)^2 = a^2 + 2ab + b^2$

Example 7.4

Factorize

$9p^2 + 24pq + 16q^2$

Solution:

$9p^2 = (3p)^2$, just like a^2

$16q^2 = (4q)^2$, just like b^2

$24pq = 2(3p)(4q)$, just like $2ab$

Applying, $(a + b)^2 = a^2 + 2ab + b^2$

Factorization of

$9p^2 + 24pq + 16q^2 = (3p + 4q)^2 = (3p + 4q)(3p + 4q)$

4. Factorization of trinomials of the form $x^2 + bx + c$

Example 7.3

Factorize $x^2 + 13x + 36$

Solution:

Leading coefficient of x^2 is 1.

Set $x^2 + 13x + 36 = (x + a)(x + b)$,

Where a and b are two numbers such that

a + b = 13, and a.b = 36

to find a and b, express 36 as product of pairs of its factors.

36 = 36 x 1,

36 = 2 x 18,

36 = 3 x 12,

36 = 4 x 9

from factorization of 36 into the above four forms, in the pair 4 and 9, the sum is 13, the numerical coefficient of the middle term in $x^2 + 13x + 36$

Now,

$x^2 + 13x + 36 =$

$(x^2 + 4x) + (9x + 36)$

$= x(x + 4) + 9(x + 4)$

$= (x + 4)(x + 9)$

5. Factorization of trinomials of the form $ax^2 + bx + c$

Example 7.6

Factorize

$12x^2 - x - 1$

Solution:

$12x^2 - x - 1$

Here a = 12, b = –1 and c = –1

Now, a x c = –12 and b = –1

4 x 3 = 12.

Adjust signs of 4 and 3 so that their sum is -1. So,

$-4 + 3 = -1$

Now,

$12x^2 - x - 1 =$

$12x^2 - 4x + 3x - 1 =$

$4x(3x - 1) + 1(3x - 1) = (4x + 1)(3x - 1)$

6. Factorization of difference of two perfect squares

Example 7.7

Factorize

$9x^2 - 16y^2$

Solution:

$9x^2 = (3x)^2$ and $16y^2 = (4y)^2$

Using the algebraic formula:

$a^2 – b^2 = (a – b)(a + b)$

factorization of

$9x^2 – 16y^2 = (3x – 4y)(3x + 4y)$

{order of factors does not matter, so it is equally correct to write the product as $(3x + 4y)(3x – 4y)$}

Application activity 7.2

Factorize:

1. $ax + bx + ay + by$
2. $12x^2 + 18x^3$
3. $a^2 + bc + ab + ac$
4. $p^2 qx + pq^2y + px^2 y + qxy^2$
5. $9p^2 + 24pq + 16q^2$
6. $4x^2 + 12x + 9$
7. $x^2 + 13x + 36$
8. $-4x^2 + 12x + 9$
9. Factorize $-6x^2 + x + 1$
10. Factorize $27y^2 – 48z^2$

Roots of polynomials

We say that $x = a$ is a root or zero of a polynomial $P(x)$ if it is a solution to the equation $P(x) = 0$. In other words, $x = a$ is a root of a polynomial $P(x)$ if $P(a) = 0$. In that case $x – a$ is one of the factors of the polynomial $P(x)$.

Consider the polynomial

$P(x) = x^4 – 2x^3 + 3x^2 – 2x – 8$

If we substitute $x = 2$ into the polynomial, we obtain

$P(2) = (2)^4 – 2(2)^3 + 3(2)^2 – 2(2) – 8 = 0$

Therefore, $x = 2$ is a root of the polynomial. Note that $(x – 2)$ is a factor of

$x^4 – 2x^3 + 3x^2 – 2x – 8$

Example 7.8

Factorize the polynomial $P(x) = x^3 - 2x^2 - 5x + 6$.

Solution

The factors of 6 are ±1, ±2, ±3 and ±6

Verify if $P(1) = 0$ $P(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$. Hence, $x - 1$ is a factor of $P(x) = x^3 - 2x^2 - 5x + 6$.

Using long division, we can write $P(x) = (x - 1)Q(x)$, where

$$Q(x) = \frac{x^3 - 2x^2 - 5x + 6}{(x - 1)} = x^2 - x - 6$$

$$Q(x) = x^2 - 3x + 2x - 6 = (x^2 - 3x) + (2x - 6) = x(x - 3) + 2(x - 3) = (x - 3)(x + 2)$$

Thus, $P(x) = (x - 1)(x - 3)(x + 2)$

7.2 Numerical functions

Definition

Consider two sets A and B. A function from A into B is a rule which assigns a unique element of A exactly one element of B. We write the function from A to B as $f: A \rightarrow B$. Most of time we do not indicate the two sets and we simply write $y = f(x)$.

The sets A and B are called the **domain** and **range** of f, respectively. The domain of f is denoted by *dom* (f).

Example 7.9

Let $f(x) = \frac{5x}{(x + 1)}$. Find the following:

(a) $f(2)$

(b) $f(2x - 7)$

(c) $f(x + h)$

(d) $f(x^2)$

(e) $f(\frac{1}{x^2})$

Solution

(a) $f(2) = \frac{5(2)}{2 + 1} = \frac{10}{3}$

(b) $f(2x - 7) = \frac{5(2x - 7)}{(2x - 7) + 1} = \frac{10x - 35}{2x - 6}$

(c) $f(x + h) = \frac{5(x + h)}{x + h + 1} = \frac{5x + 5h}{x + h + 1}$

(d) $f(x^2) = \frac{5x^2}{x^2 + 1}$

(e) $f(\frac{1}{x^2}) = \frac{5(\frac{1}{x^2})}{(\frac{1}{x^2}) + 1} = \frac{\frac{5}{x^2}}{\frac{1+x^2}{x^2}} = \frac{5}{1 + x^2}$

Domain and range of a function

We have been introduced to functions. These involve the relationship or rule connecting domains to ranges.

Example 7.10

State the domain and range of the following relations. Is the relation a function?
{(2, –3), (4, 6), (3, –1), (6, 6), (2, 3)}

Solution

The above list of points, being a relationship between certain *x*'s and certain *y*'s, is a relation. The domain is all the *x*-values, and the range is all the *y*-values. To give the domain and the range, just list the values without duplication:

domain: {2, 3, 4, 6}

range: {–3, –1, 3, 6}

While the given set does represent a relation (because *x*'s and *y*'s are being related to each other), they give us two points with the same *x*-value: (2, –3) and (2, 3). Since $x = 2$ gives me two possible destinations, then **this relation is not a function.**

Note that we have to check whether the relation is a function by looking for duplicate *x*-values. If you find a duplicate *x*-value, then the different *y*-values mean that you do *not* have a function.

Domain

Domain of a function is the set of all real numbers for which the expression of the function is defined as a real number. In other words, it is all the real numbers for which the expression makes sense.

Determination of domain of definition of a function in $\mathbb{R}$

When determining domain of definition, we have to take into account three things:

- You cannot have a zero in the denominator.
- You cannot have a negative number under an even root.
- You cannot evaluate the logarithm of a negative number

Example 7.11

Find the domain of the function of each of the following:

1. $f(x) = 3x^3 - 2x^2 + 4x - 6$
2. $g(x) = \frac{2x + 3}{x - 3}$
3. $h(x) = \sqrt{6 - 3x}$

Solution

1. The function $f(x) = 3x^3 - 2x^2 + 4x - 6$ is defined for all real numbers x. Therefore, the domain of f is $\mathbb{R}$.
2. The function $g(x) = \frac{2x + 3}{x - 3}$ is defined for all real numbers except the number that makes the denominator to be zero. We set $x - 3 \neq 0$ which gives us $x \neq 3$. Therefore, the domain of g is $\{x \in \mathbb{R}: x \neq 3\} = \mathbb{R} \setminus \{3\}$.
3. We know that the square root of the negative number does not exist in $\mathbb{R}$. The function $h(x) = \sqrt{6 - 3x}$ is defined if and only if

   $6 - 3x \geq 0$

   $-3x \geq -6$

   $3x \leq 6$

   $x \leq 2.$

   Therefore, the domain of the function h is $\{x \in \mathbb{R}: x \leq 2\} = (-\infty, 2]$.

Range

Let $f: A \to B$ be a function. The range of f, denoted by Im(f) is the image of A under f, that is, Im (f) = f[A]. The range consists of all possible values the function f can have.

Steps to finding range of function

To find the range of function f described by formula where the domain is taken to be the natural domain:

1. Put $y = f(x)$.
2. Solve x in terms of y
3. The range of f is the set of all real numbers y such that x can be solved.

Example 7.12

For each of the following functions, find the range.

1. $f(x) = x + 5$
2. $g(x) = \frac{8}{2x - 4}$
3. $h(x) = \sqrt{6 - 3x}$

Solution

(1) Put $y = f(x) = x + 5$

Solve for x; $x = y - 5$

Note that x can be solved for any value of y. Therefore, the range is $\mathbb{R}$.

(2) Put $y = g(x) = \frac{8}{2x - 4}$.

Solve for x; $(2x - 4)y = 8$

$2xy - 4y = 8$

$2xy = 8 + 4y$

$x = \frac{8 + 4y}{2y}$

Note that x can be solved if and only if $2y \neq 0$. i.e. $y \neq 0$.

Therefore, the range of the function g is $\{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} \setminus \{0\}$.

(3) Put $y = h(x) = \sqrt{6 - 3x}$. We see that $y \geq 0$.

Solve for x; $y = \sqrt{6 - 3x}$, $y \geq 0$

$y^2 = 6 - 3x$; $y \geq 0$

$x = \frac{6 - y^2}{3}$; Here x can be solved whenever $y \geq 0$.

Therefore, the range of the function h is $\{y \in \mathbb{R}: y \geq 0\} = [0, \infty)$

Example 7.13

Determine the domain and range of the given function:

Solution

$$y = \frac{x^2 + x - 2}{x^2 - x - 2}$$

The domain is a set of all the values that *x* is allowed to take. The only problem we have with this function is that we need to be careful not to divide by zero. So the only values that *x* cannot take on are those which would cause division by zero. So we set the denominator equal to zero and solve:

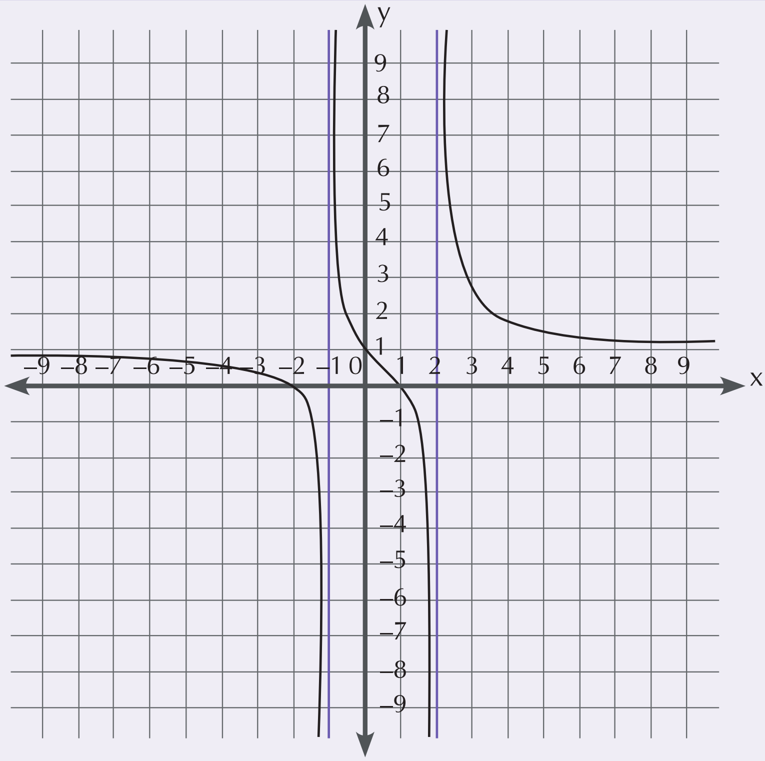


Fig. 7.1

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

Then the domain is "all x not equal to –1 or 2".

As we can see from, the graph "covers" all *y*-values (that is, the graph will go as low as I like, and will also go as high as I like). Since the graph will eventually cover all possible values of y, then the range is "all real numbers".

Example 7.14

Determine the domain and range of the given function:

$y = -\sqrt{-2x + 3}$

The domain is all values that *x* can take on. The only problem we have with this function is that we cannot have a negative inside the square root. So we set the insides greater-than-or-equal-to zero, and solve.

$-2x + 3 \geq 0$

$-2x \geq -3$

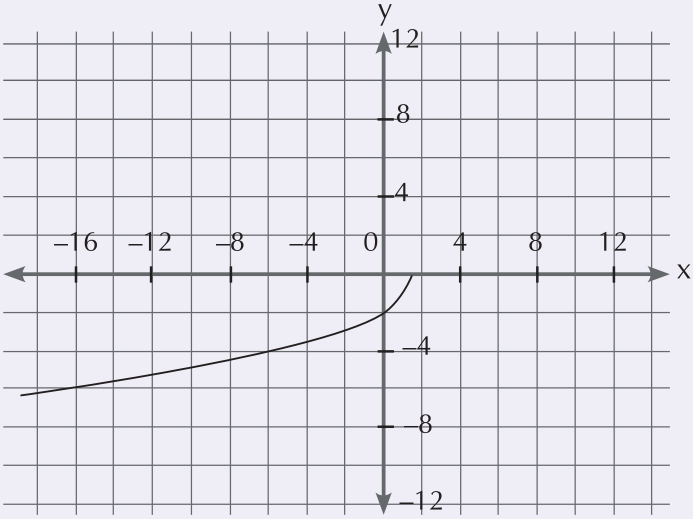


Fig. 7.2

$2x \leq 3$

$x \leq \frac{3}{2} = 1.5$

Then **the domain is "all x ≤ $\frac{3}{2}$".** $Range = \left]-\infty, 0\right]$

Example 7.15

Determine the domain and range of the given function:

$y = -x^4 + 4$

Solution

This is just a type of polynomial. There are no denominators (so no division-by-zero problems) and no radicals (so no square-root-of-a-negative problems). There are no problems with a polynomial. There are no values that we cannot substitute for *x*. When I have a polynomial, the answer is always that **the domain is "all x". Dom=** $\mathbb{R}$

$Range = \left]-\infty, 4\right]$

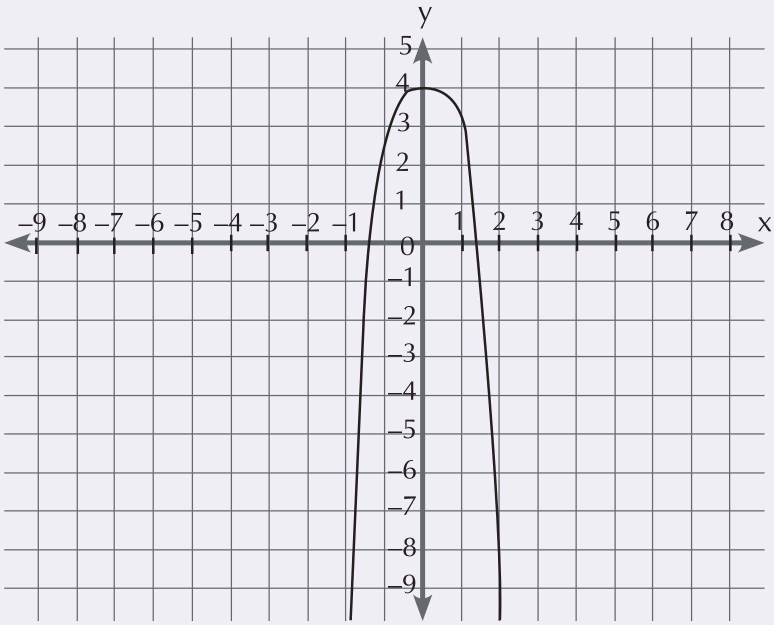


Fig. 7.3

Application activity 7.2

1. Find the domain of f in each of the following.

   (a) $f(x) = \sqrt{x + 7} - \sqrt{x^2 + 2x - 15}$

   (b) $f(x) = x^5 - 3x^4 + x^2 - 5$

   (c) $f(x) = \frac{7x^2 + 9}{x + 6}$

   (d) $f(x) = \sqrt{2x - 3}$

   (e) $f(x) = \frac{x^2 + 1}{x^2 - x - 2}$

   (f) $f(x) = \frac{\sqrt{x}}{4x - 7}$

   (g) $f(x) = \frac{2x - 3}{\sqrt[4]{x - 7}}$

   (h) $f(x) = \sqrt{\frac{x + 1}{x - 1}}$

   (i) $f(x) = \sqrt{x^2 - 5x + 6}$

   (j) $f(x) \frac{\sqrt{2 + 7x}}{x^2 - 8x + 7}$

   (k) $f(x) = \frac{3x^2 + 5x - 2}{\sqrt[3]{7 - 3x}}$

2. Let $f(x) = \frac{2x + 1}{x^2 + 1}$. Find the range of f.

3. Find the range of each of the following functions.

   a) $y = 2x - 3$ for $x \geq 0$

   b) $y = x^2 - 5$ for $x \leq 0$

   c) $f(x) = 1 - x$ for $x \leq 1$

   d) $f(x) = \frac{1}{x}$ for $x \geq 2$

4. The function f is such that $f(x) = -x$ for $x < 0$, and $f(x) = x$ for $x \geq 0$.

Find the values of:

a) f(5)

b) f(–4)

c) f(–2)

d) f(0)

Composition of functions

The composition of g and f, denoted $g \circ f$ is defined by the rule $(g \circ f)(x) = g(f(x))$ provided that f(x) is in the domain of g.

Note: $f \circ g$ is also read as "f compose g," and $g \circ f$ is also read as "g compose f."

The functions $g \circ f$ and $f \circ g$ are called the **composite** or **compound** functions and in general, $g \circ f \neq f \circ g$.

Example 7.16

Given the functions defined by $f(x) = 4x + 3$ and $g(x) = 7x$. Find

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(c) $(f \circ g)(2)$

(d) $(g \circ f)(2)$

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(7x) = 4(7x) + 3 = 28x + 3$

(b) $(g \circ f)(x) = g(f(x)) = g(4x + 3) = 7(4x + 3) = 28x + 21$

(c) $(f \circ g)(2) = 28(2) + 3 = 56 + 3 = 59$

(d) $(g \circ f)(2) = 28(2) + 21 = 56 + 21 = 77$

Application activity 7.4

1. Find f(g(x)) if:

   (a) $f(x) = x^2$ and $g(x) = 2x + 7$

   (b) $f(x) = 2x + 7$ and $g(x) = x^2$

   (c) $f(x) = \sqrt{x}$ and $g(x) = 3 - 4x$

   (d) $f(x) = 3 - 4x$ and $g(x) = \sqrt{x}$

   (e) $f(x) = \frac{2}{x}$ and $g(x) = x^2 + 3$

   (f) $f(x) = x^2 + 3$ and $g(x) = \frac{2}{x}$

2. Find f(x) and g(x) given that f(g(x)) is:

   a) $(3x + 10)^3$

b) $\frac{1}{2x + 4}$

c) $\sqrt{x^2 - 3x}$

d) $\frac{10}{(3x - x^2)^3}$

The inverse of a numerical function

For a one-to-one function defined by y = f(x), the equation of the inverse can be found as follows:

I Replace f(x) by y.

II Interchange x and y.

III Solve for y.

IV Replace y by $f^{-1}(x)$

Example 7.17

Find the inverse of the one-to-one functions defined by:

(a) $f(x) = 3 - x^3$

(b) $g(x) = \frac{2x + 3}{x - 1}$

Solution

(a) $y = 3 - x^3$

$x = 3 - y^3$

$x - 3 = -y^3 \Leftrightarrow y^3 = 3 - x \Leftrightarrow y = \sqrt[3]{3 - x}$

$f^{-1}(x) = \sqrt[3]{3 - x}$

(b) $g(x) = \frac{2x + 3}{x - 1}$

$y = \frac{2x + 3}{x - 1}$

$x = \frac{2y + 3}{y - 1}$

$xy - x = 2y + 3 \Leftrightarrow xy - 2y = 3 + x \Leftrightarrow y\,(x - 2) = x + 3 \Leftrightarrow y = \frac{x + 3}{x - 2}$

$f^{-1}(x) = \frac{x + 3}{x - 2}$

Note: $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

Example 7.18

Determine whether $f(x) = \frac{1}{x-3}$ and $g(x) = \frac{3x+1}{x}$ are inverse functions by computing their compositions.

Solution

Since $(f \circ g)(x) = f(g(x)] = f\left(\frac{3x+1}{x}\right) = \frac{1}{\frac{3x+1}{x} - 3} = \frac{1}{\frac{3x+1-3x}{x}} = \frac{1}{\frac{1}{x}} = x$ and

$$(g \circ f)(x) = g(f(x)] = g\left(\frac{1}{x-3}\right) = \frac{3\left(\frac{1}{x-3}\right)+1}{\frac{1}{x-3}} = \frac{\frac{3+x-3}{x-3}}{\frac{1}{x-3}} = \frac{\frac{x}{x-3}}{\frac{1}{x-3}} = x$$

Thus the functions f and g are a pair of inverse functions.

Parity of a function

Even function

Let f be a function of $\mathbb{R}$ in $\mathbb{R}$ we say that f is **even** if $\forall\, x \in$ Dom (f), $(-x) \in$ Dom(f); $f(-x) = f(x)$

For example:

$f: \mathbb{R} \to \mathbb{R}: x \to f(x) = x^2$ is an even function because

$f(-x) = (-x)^2 = x^2 = f(x)$

Note: The graph of an even function is symmetric with respect to the y-axis.

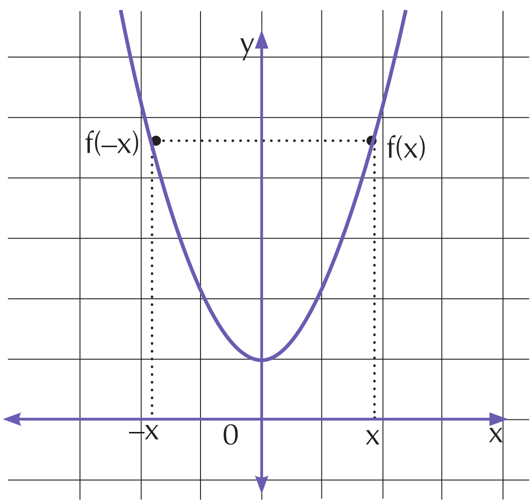


Fig. 7.4

This is the curve $f(x) = x^2+1$

Odd function

We say that a function f is **odd** if $\forall x \in$ Dom (f), $(-x) \in$ Dom(f);

$f(-x) = -f(x)$

For example:

$f: \mathbb{R} \to \mathbb{R}: x \to f(x) = x^3$ is odd function because

$f(-x) = (-x)^3 = -x^3 = -f(x)$

And we get an origin symmetry:

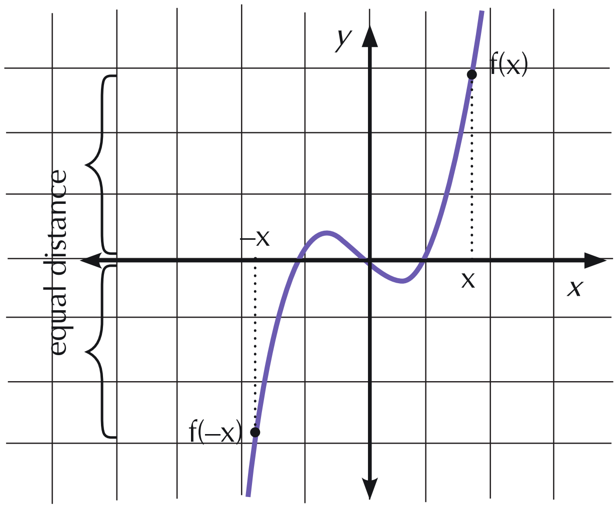


Fig. 7.5

This is the curve $f(x) = x^3 - x$

Application activity 7.5

State whether the following functions are even or odd

1. $f(x) = x^2 - 1$
2. $g(x) = \frac{x^3 + x}{5}$
3. $h(x) = \frac{x^3 + x + 2}{x^2}$

7.3 Application of rational and irrational functions

Activity 7.3

In groups of five, research on the different ways of applying functions of rational and irrational number. Discuss your findings in class. Have at least one person from each group demonstrate an example of application to the rest of the class.

Free falling objects

A free falling object is accelerated toward the earth with a constant acceleration equal to $g = 9.8\ m/s^2$. The different formulas used in dealing with free falling objects are:

1. $y = y_0 + v_0t + \frac{1}{2}gt^2$ (Quadratic or polynomial equation)
2. $v = v_0 + gt$

An equation relating v, g and y but not containing the time t is:

$v^2 - v_0^2 = 2g(y - y_0)$

Where g: acceleration due to gravity, v_0: initial velocity, v: final velocity, y_0: the initial height and y: the final height.

Example 7.19

Mukansanga throws a ball vertically upward and it reaches the height of 4.9m. Find

a) the time taken by the ball to reach the maximum height

b) the initial speed by which the ball was thrown.

Solution

We know that the equation of the moving ball is the quadratic equation

$y = v_0t + \frac{1}{2}at^2$.

The acceleration of the ball is $a = -g = -9.8\ m/s^2$.

(a) The time taken is found by solving the quadratic equation $y = v_0t - \frac{1}{2}gt^2$, where $y = 4.9$, $v = v_0 - gt$ $\quad v_0 = v + gt$ and $v = 0$ $v_0 = gt$

$4.9 = v_0t - \frac{1}{2}(9.8)t^2$ where $\quad v_0 = gt$

$4.9 = (gt)t - \frac{1}{2}(9.8)t^2$

$4.9 = 9.8t^2 - 4.9t^2$

$4.9 = 4.9t^2$

$1 = t^2$

$t = \pm 1$

Since the negative time is ruled out, the time taken to reach the maximum height is 1s.

b) The initial speed is found by solving the linear equation $v = v_0 - gt$ where $v = 0$ and

$t = 1$.

$v_0 = gt$

$v_0 = 9.8(1) = 9.8\ m/s$

The initial speed by which the ball was thrown is 9.8 m/s.

Projectiles

We consider the motion of a projectile fired from ground level at an angle θ upward from the horizontal and an initial speed v_0. We are interested in finding, how long it takes the projectile to reach the maximum height, the maximum height the projectile reaches and the horizontal distance it travels (called the range, R). In general we use the following formulas:

$$y = (v_0 \sin \theta)t - \frac{1}{2}gt^2$$

The projectile reaches maximum height when its y-velocity is zero

$(v_0 \sin \theta)t - gt = 0 \qquad t = \frac{v_0 \sin \theta}{g}$

The maximum height is reached by the projectile in time when $t = \frac{v_0 \sin \theta}{g}$

$$H = (v_0 \sin \theta)\left(\frac{v_0 \sin \theta}{g}\right) - \frac{1}{2}\left(\frac{v_0 \sin \theta}{g}\right)^2 \qquad H = \frac{v_0^2 \sin^2 \theta}{2g}$$

At the time when the projectile hits the same horizontal level y = 0 and we have to solve the quadratic equation and $y = (v_0 \sin\theta)\, t - \frac{1}{2} gt^2 = 0$

$$t(v_0 \sin \theta - \frac{1}{2} gt) = 0 \qquad t = 0 \text{ or } t = \frac{2v_0 \sin \theta}{g}$$

The range is the distance travelled by the projectile in time $t = \frac{2v_0 \sin \theta}{g}$

$$x = (v_0 \cos \theta)\, t - \frac{1}{2} a_x t^2 = (v_0 \cos \theta)\left(\frac{2v_0 \sin \theta}{g}\right) - 0 = \frac{2v_0^2 \sin \theta \cos \theta}{g}$$

$$R = \frac{2v_0^2 \sin \theta \cos \theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

Example 7.20

A ball is kicked with an initial velocity of 25m/s at an angle of 60° above the horizontal ground. Find:

a) the maximum height reached by the ball (H)

b) the time taken by the ball to reach the maximum height

c) the horizontal distance reached by the ball from the point of the kick to the point it hits the ground (range R).

At what angle should the ball be kicked so that it reaches the longest distance along the ground?

Solution

a) The maximum height reached by the ball is

$$H = \frac{v_0^2 \sin^2 \theta}{2g} = \frac{(25)^2 \sin^2 60}{2(9.8)} = \frac{625\left(\frac{\sqrt{3}}{2}\right)^2}{19.6} = \frac{1875}{4} \times \frac{1}{19.6} = \frac{1875}{78.4} = 23.92\text{m}$$

b) The time taken by the ball to reach the maximum height is

$$t = \frac{V_0 \sin\theta}{g} = \frac{25 \sin 60}{9.8} = \frac{25\left(\frac{\sqrt{3}}{2}\right)}{9.8} = \frac{25\sqrt{3}}{2(9.8)} = 2.21s.$$

c) The range is $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(25)^2 \sin(120)}{9.8} = \frac{541.27}{9.8} = 55.23m.$

In the formula $\frac{v_0^2 \sin 2\theta}{g} = \frac{625 \sin 2\theta}{9.8}$ $\sin 2\theta = 1$ $2\theta = 90°$ $\theta = 45°$

The ball should be kicked at an angle $\theta = 45°$.

Determination of the equilibrium cost (price) of a commodity in an isolated market

We have three variables:

Q_d: the quantity demand of the commodity

Qs: the quantity supplied of the commodity

P: the (cost) price of the commodity.

The equilibrium condition is $Q_d = Q_s$

The model is

$Q_d = Q_s$, where

$Q_d = -ap + b$ $(a, b > 0)$

$Q_s = cp - d$ $(c,d > 0)$

We can find the equilibrium price by successive elimination of equations and variables through substitution.

From $Q_d = Q_s$, we have

$-ap + b = cp - d$

$(c + a)p = b + d$

$$p = \frac{b + d}{c + a}$$

Thus, the equilibrium cost (price) of commodity is

$$\overline{p} = \frac{b + d}{c + a}$$

The equilibrium quantity can be obtained by substituting $\overline{p}$ into either Q_s or Q_d;

$$\overline{Q} = -a\left(\frac{b + d}{c + a}\right) + b = \frac{-a(b + d) + b(c + a)}{c + a} = \frac{-ab - ad + bc + ab}{c + a} = \frac{bc - ad}{c + a}.$$

Since the denominator $(c + a)$ is positive, the positivity of $\overline{Q}$ requires that the numerator $(bc - ad) > 0$. Thus, to be economically meaningful, the model should contain the additional restriction that $bc > ad$.

Example 7.21

Below is the model of an isolated market for a commodity x.

$Q_d = -x + 6$

$Q_s = 4x - 4$

Find:

a) the equilibrium cost of the commodity for the model below

b) the equilibrium quantity of a commodity.

Solution

a) In equilibrium state, $Q_d = Q_s$

$-x + 6 = 4x - 4$

$5x = 10$

$x = 2$

Thus, the equilibrium cost of commodity is **2**

b) When the cost is 2, then the quantity is $-2 + 6 = 4$.

Thus, the equilibrium quantity of a commodity is **4**.

Example 7.22

Find the equilibrium cost of the commodity for the following model

$Q_d = 15 - x^2$

$Q_s = 4x - 6$

Solution

In equilibrium state, $Q_d = Q_s$

$15 - x^2 = 4x - 6$

$x^2 + 4x - 21 = 0$

We can find x using the formula for the roots of the quadratic equation

$ax^2 + bx + c = 0$ which is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-4 \pm \sqrt{16 + 84}}{2} = \frac{-4 \pm \sqrt{100}}{2} = \frac{-4 \pm 10}{2}$$

$x = 3$ or $x = -7$

Since the negative prices are ruled out, the only equilibrium cost of the commodity is **3**.

Rates of reaction in chemistry

The rate of reaction of a chemical reaction is the rate at which products are formed. To measure a rate of reaction, we simply need to examine the concentration of one of the products as a function of time.

Example 7.23

Calculate the rate constant for the reaction between phenolphthalein and the OH^- ion if the instantaneous rate of reaction is 2.5×10^{-5} mole per litre per second when the concentration of phenolphthalein is 0.0025 M.

Solution

We start with the rate law for this reaction:

Rate = k(phenolphthalein)

We then substitute the known rate of reaction and the known concentration of phenolphthalein into this equation:

Solving for the rate constant gives the following result:

$k = 0.010\ s^{-1}$

Application activity 7.6

1. The weekly sale *S* (in thousands of units) for the t^{th} week after the introduction of the product in the market is given by $S = \frac{120t}{t^2 + 100}$. In which week would the sale (*S*) have been 6?
2. After taking a certain antibiotic, the concentration (C) of the drug in the patient's bloodstream is given by $C(x) = \frac{0.04t}{t^2 - 2}$, where *t* is the time (in hours) after taking the antibiotic. How many hours after taking the antibiotic will its concentration be 0.04 units?
3. The average cost per unit C(x), in Rwandan Francs, to produce *x* units of matchboxes is given by $C(x) = \frac{300}{x + 10}$. What is the cost per unit when 600 matchboxes are produced?

Summary

1. A function f given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, is called a polynomial function of degree n.
2. Factorization is a way of writing a polynomial as a product of two or more factors.
3. We say that $x = a$ is a root or zero of a polynomial $P(x)$ if it is a solution to the equation $P(x) = 0$.
4. Domain of a function is the set of all real numbers for which the expression of the function is defined as a real number.
5. The composition of functions g and f, denoted $g \circ f$ is defined by the rule $(g \circ f)(x) = g(f(x))$ provided that $f(x)$ is in the domain of g.
6. For a one-to-one function defined by $y = f(x)$, the equation of the inverse can be found as follows:
   I. Replace $f(x)$ with y.
   II. Interchange x and y.
   III. Solve for y.
   IV. Replace y with $f^{-1}(x)$.

Topic area: Analysis

Sub-topic area: Limits, differentiation and integration

Unit 8

Limits of polynomial, rational and irrational functions

Key unit competence

Evaluate correctly limits of functions and apply them to solve related problems.

8.0 Introductory activity

To find the value of a function $f(x)$ when x approaches 2, a student used a calculator and dressed a table as follows:

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

a) Is it possible to put the values of x on a number line? Try to do it and locate the point $x=2$

b) Write 2 possible open intervals of the number line such that their centre is $x=2$

c) Try to approximate the value of $f(x)$ when x approaches 2.

8.1 Concept of limits

8.1.1 Neighbourhood of a real number

Activity 8.1

You have heard about the term 'neighbourhood' in everyday life. What does it mean?

Carry out research to find the meaning of the term in mathematics. Discuss your findings in class. Use diagrams in your explanations.

By a neighbourhood of a real number c we mean an interval which contains c as an interior point.

A neighbourhood of c

On the real line, a neighborhood of a real number is an open interval $(a - \delta, a + \delta)$ where $\delta > 0$, with its centre at a. Therefore, we can say that a neighborhood of the real number is any interval that contains a real number a and some point below and above it.

We can add:

Let x be a real number. A neighbourhood of x is a set $\mathbb{N}$ such that for some $\varepsilon > 0$ and for all y, if $|x - y| < \varepsilon$ then $y \in \mathbb{N}$.

8.1.2 Limit of a variable

Activity 8.2

In groups of five, work out the following:

Let $f(x) = 2x + 2$ and compute f(x) as x takes values closer to 1. First consider values of x approaching 1 from the left ($x < 1$) then consider x approaching 1 from the right ($x > 1$).

In Activity 8.2, x approaching 1 from the left ($x < 1$) gives us:

x	f(x)
0.5	3
0.8	3.6
0.9	3.8
0.95	3.9
0.99	3.98
0.999	3.998
0.9999	3.9998
0.99999	3.99998

and x approaching 1 from the right ($x > 1$) gives us:

x	f(x)
1.5	5
1.2	4.4
1.1	4.2
1.05	4.1
1.01	4.02
1.001	4.002
1.0001	4.0002
1.00001	4.00002

In both cases as x approaches 1, f(x) approaches 4. Intuitively, we say that $\lim_{x\to 1} f(x) = 4$.

Note: We are talking about the values that f(x) takes when x gets closer to 1 and not f(1). In fact we may talk about the limit of f(x) as x approaches a even when f(a) is undefined.

Suppose the function y = f(x) is a numerical function with independent variable x and dependent variable y where the value y depends on the variable of x. If the values of x can be made as closer to a value a as we please, then this can be written $x \to a$. This is read as 'x tends to a' or ' x approaches a '.

Since the variable y depends on the variable x , we can say that as x tends to a, y will also tend to a certain value that we have to determine. The obtained value of y as x tends to a is the what we call the **limiting value** of f(x).

Suppose that f(x) = 2x + 1. Let us find the limiting value of f(x) as x tends to 3.

From the left of 3

x	2	2.5	2.9	2.99	2.999
f(x)	5	6	6.8	6.98	6.998

We say that as x approaches 3 from the left, f(x) approaches 7 from below.

From the right

x	4	3.5	3.1	3.01	3.001
f(x)	9	8	7.2	7.02	7.002

In this case we say that as x approaches 3 from the right, f(x) approaches 7 from above.

From either side

x	2	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5	4
f(x)	5	6	6.8	6.98	6.998	L	7.002	7.02	7.2	8	9

In summary, we can see that as x approaches 3 from either direction, f(x) approaches a value of 7, and we write

$$\lim_{x\to 1} f(x) = \lim_{x\to 3} 2x + 1 = 7$$

If f(x) can be made as close as we like to some number a by making x sufficiently close to (but not equal to) a, then we say that f(x) has a limit of L as x approaches a, and we write

$$\lim_{x\to a} f(x) = L$$

Note that in the above case the limit is found by the direct substitution i.e.

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} 2x + 1 = 2(3) + 1 = 7 = f(3).$$

Sometimes the direct substitution may appear to give an indeterminate case like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ ... and we have to first transform or simplify the function.

Example 8.1

Evaluate the following limits.

1. $\lim_{x \to 2} x^3 - 3x^2 + 4x + 5$
2. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$
3. $\lim_{x \to 0} \frac{x^2 - 3x}{x}$
4. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Solution

1. $\lim_{x \to 2} x^3 - 3x^2 + 4x + 5 = (2)^3 - 3(2)^2 + 4(2) + 5 = 8 - 12 + 8 + 5 = 9$
2. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$, direct substitution cannot be applied and we can simplify first.

   $$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{x - 1} = \lim_{x \to 1} x + 2 = 1 + 2 = 3$$

3. $\lim_{x \to 0} \frac{x^2 + 3x}{x} = \lim_{x \to 0} \frac{x(x+3)}{x} = \lim_{x \to 0} (x + 3) = 3$
4. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x + 3)(x - 3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6$

Note: If we have to find $\lim_{x \to a} f(x)$, the function does not have to be defined at a point at which the limit is taken since we are only concerned with what happened around the point. The function must be defined in the neighbourhood of a .

The following graph shows that as x approaches 1 from the left, y = f(x) approaches 2 and this can be written as

$$\lim_{x \to 1^-} f(x) = 2$$

As x approaches 1 from the right, y = f(x) approaches 4 and this can be written as

$$\lim_{x \to 1^+} f(x) = 4$$

Note that the left and right hand limits and f(1) = 3 are all different.

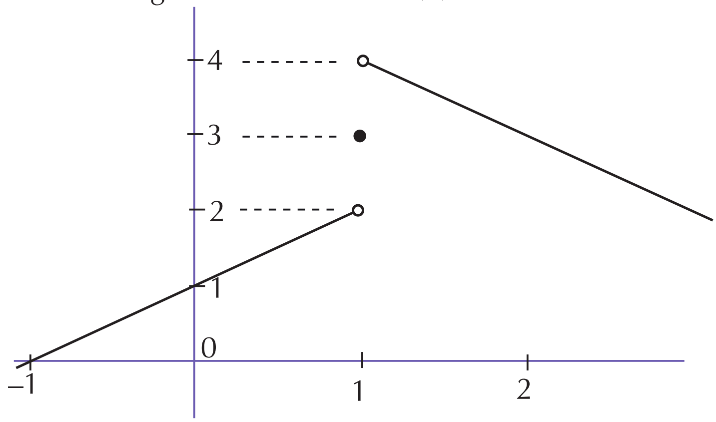


Fig. 8.1

Application activity 8.1

1. What is $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$?

2. What is $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x + 3}$?

3. Draw the graph of $g(x) = \dfrac{\sin x}{x}$ and compute g(x) as x takes values closer to 0. **Hint**: consider values of x approaching 0 from the left ($x < 0$) and values of x approaching 0 from the right ($x > 0$).

4. The function f is defined by:

   $f(x) = x - 2$ for $x \le 1$

   $f(x) = x^2$ for $x > 1$

   Use graphical method to find:

   $\lim\limits_{x \to 1} f(x)$.

One-sided limits

Right-sided limits

If x is taking values sufficiently close and greater than a , then we say that x tends to a from above and the limiting value is then what we call the right-sided limit.

$$\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a \\ >}} f(x)$$

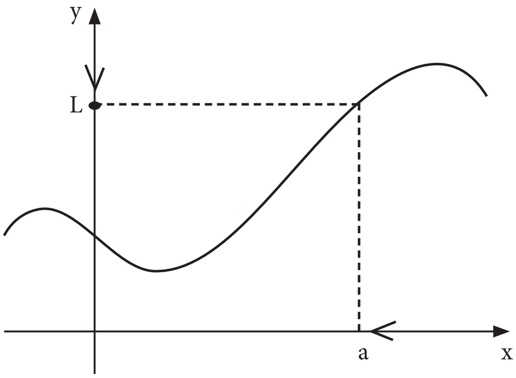


Fig. 8.2

Let $a \in \mathbb{R}$ and let f be a function such that f(x) is defined for x sufficiently close to and greater than a. Suppose L is a real number satisfying f(x) is arbitrarily close to L if x is sufficiently close to and greater than a.

Then we say that L is the right-sided limit of f at a and we write $\lim\limits_{x \to a+} f(x) = L$.

Left-sided limits

If x is taking values sufficiently close to and less than a, then we say that x tends to a from below and the limiting value is then what we call the left-sided limit. It is written as:

$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a \\ <}} f(x)$$

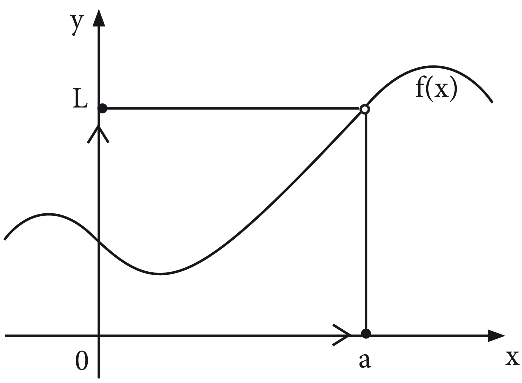


Fig. 8.3

If a function f is defined on the left-side of a , we can consider its left-side limit. The notation $\lim\limits_{x \to a-} f(x) = L$ means that f(x) is arbitrarily close to L if x is sufficiently close to and less than a.

Then we say that L is the left-side limit of f at a and we write $\lim\limits_{x \to a} f(x) = L$.

Two-sided limits

In writing $\lim_{x \to a} f(x) = L$, this means that the function f(x) tends to L as x tends to a from the either side (left and right). Thus, the two-sided limit exist if the two limits (left and right) exist and are equal.

If $\lim_{x \to a-} f(x) = L = \lim_{x \to a+} f(x)$ then we have $\lim_{x \to a} f(x) = L$ and

If $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$ then $\lim_{x \to a} f(x)$ does not exist.

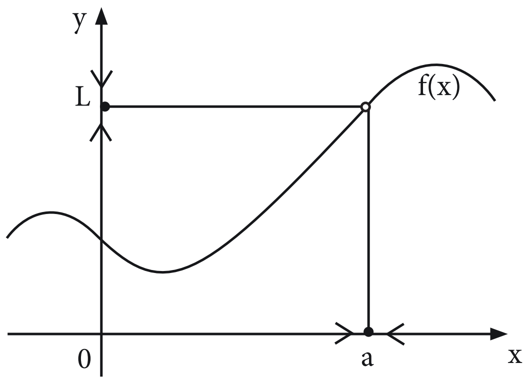


Fig. 8.4

Definition

If the f(x) tends closer to a value L as x approaches the value a from either side, then L is the limit of f(x) as x approaches a . We use the following notation: $\lim_{x \to a} f(x) = L$.

Example 8.2

Find $\lim_{x \to 0} \left(x + \frac{|x|}{x}\right)$

Solution

Since $x + \frac{|x|}{x} = \begin{cases} x + 1 \text{ for } x \geq 0 \\ x - 1 \text{ for } x < 0 \end{cases}$

$\lim_{x \to 0} \left(x + \frac{|x|}{x}\right) = \lim_{x \to 0} (x + 1) = 1$

$\lim_{x \to 0} \left(x + \frac{|x|}{x}\right) = \lim_{x \to 0} (x - 1) = -1$

Thus, $\lim_{x \to 0} \left(x + \frac{|x|}{x}\right)$ does not exist because $\lim_{x \to 0} \left(x + \frac{|x|}{x}\right) = 1$ which is not equal to $\lim_{x \to 0} \left(x + \frac{|x|}{x}\right) = -1$

Example 8.3

Let $f(x) = \frac{|x|}{x}$. Find $\lim_{x \to 0} f(x)$

Solution

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = \lim_{x \to 0^-} -1 = -1$$

Therefore, $\lim_{x \to 0^+} \frac{x}{|x|}$ does not exist.

Rules for limits of functions at a point

1. $\lim_{x \to a} k = k$ (where $a \in \mathbb{R}$ and k is a constant)
2. $\lim_{x \to a} x^n = a^n$ (where $a \in \mathbb{R}$ and n is a positive integer)
3. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (where $a \in \mathbb{R}$ and n is an odd positive integer)
4. $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

   The result is valid for sum and difference of finitely many functions.
5. $\lim_{x \to a} (f(x) \times g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$

   The result is valid for product of finitely many functions.
6. $\lim_{x \to a} \frac{f(x)}{g(x)} \quad \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided that $g(x) \neq 0$

Example 8.4

Evaluate the following limits and justify each step.

a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution

a) $\lim_{x\to 5}(2x^2 - 3x + 4) = \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4$

$= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 = 2(5^2) - 3(5) + 4$

$= 50 - 15 + 4 = 39$

b) $\lim_{x\to -2}\dfrac{x^3 + 2x^2 - 1}{5 - 3x} = \dfrac{\lim_{x\to -2}(x^3 + 2x^2 - 1)}{\lim_{x\to -2}(5 - 3x)} = \dfrac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}$

$= \dfrac{(-2)^3 + 2(-2)^2 - 1}{5-3(-2)} = \dfrac{-8 + 8 - 1}{5 + 6} = -\dfrac{1}{11}$

Evaluation of algebraic limits by direct substitution

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to -3} f(x) = f(a)$

We can directly put the limiting value of x in the function, provided the function does not assume an indeterminate form.

Example 8.5

Evaluate the following limits

(a) $\lim_{x\to 3} 5x$ (b) $\lim_{x\to 4}\dfrac{3x}{2}$ (c) $\lim_{x\to -2} x^3$

Solution

(a) $\lim_{x\to 3} 5x = 5(3) = 15$

(b) $\lim_{x\to 4}\dfrac{3x}{2} = \dfrac{3(4)}{2} = \dfrac{12}{2} = 6$

(c) $\lim_{x\to -2} x^3 = (-2)^3 = -8$

Infinity limits

Let f be a function defined on both sides of a , except possibly at a itself. Then

$\lim_{x\to a} f(x) = \infty$

means that the values of f(x) can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a.

Limits at infinity

Let f be a function defined on some interval, (a, ∞) . Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of f(x) can be made arbitrarily close to L by taking x sufficiently large.

Let f be a function defined on some interval, $(-\infty, a)$. Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of f(x) can be made arbitrarily close to L by taking x sufficiently large negative.

Consider for example a polynomial of the 3rd degree in x.

$$\lim_{x\to\infty} (ax^3 + bx^2 + xc + d) = \lim_{x\to\infty} x^3 \left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right) = a \lim_{x\to\infty} x^3.$$

This reasoning is valuable for a polynomial with any degree.

Example 8.6

Evaluate the following limit $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Solution

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{x^2\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{x^2\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} = \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$\frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{2}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{4}{x} + \lim_{x\to\infty} \frac{1}{x^2}} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$

Note:

$$\lim_{x\to\pm\infty} (ax + b) = \pm\infty \text{ if } a > 0$$

$$\lim_{x\to\pm\infty} (ax + b) = \mp\infty \text{ if } a < 0$$

$$\lim_{x\to\pm\infty} (ax^2 + bx + c) = +\infty \text{ if } a > 0$$

$$\lim_{x\to\pm\infty} (ax^2 + bx + c) = -\infty \text{ if } a < 0$$

Note: To compute the limit of a function as $x \to \pm\infty$, we use the following principles

$\forall a \in \mathbb{R}$

$$a(\pm\infty) = \begin{cases} \pm\infty \text{ if } a > 0 \\ \mp\infty \text{ if } a < 0 \end{cases},$$

$$-\infty \pm a = -\infty \text{ and } +\infty \pm a = +\infty,$$

$$\frac{a}{\pm\infty} = 0,$$

$$\frac{a}{0} = +\infty \text{ or } -\infty.$$

8.2 Theorems on limits

Compatibility with order

If for each 'x' so close to 'a', $f(x) \le g(x)$ and $\lim\limits_{x \to a} f(x) = b$, $\lim\limits_{x \to a} g(x) = c$ then $b \le c$, i.e. $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$.

Squeeze theorem

This is also known as the sandwich theorem, comparison theorem or vice theorem. If a function can be squeezed (sandwiched) between two other functions, each of which approaches the same limit b as $x \to a$, then the squeezed function also approaches the same limit as $x \to a$.

Let f(x), g(x) and h(x) be three functions such that $f(x) \le g(x) \le h(x)$.

If for each x so close to a, $\lim\limits_{x \to a} f(x) = b = \lim\limits_{x \to a} h(x)$ then the function g(x) has also a limit at a and $\lim\limits_{x \to a} g(x) = b$.

Example 8.7

Compute the following limits:

$$\lim_{x \to 0} x^2 \sin\frac{1}{x}$$

Solution

We apply the Squeeze theorem, and so we need to find a function f(x) smaller than $g(x) = x^2 \sin\frac{1}{x}$ and a function h(x) bigger than g(x) such that both f(x) and h(x) approach 0. To do this we use knowledge of the sine function. Because the sine of any number lies between –1 and 1, we can write $-1 \le \sin\frac{1}{x} \le 1$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

We know that $\lim_{x \to 0} -x^2 = 0$ and $\lim\limits_{x \to 0} x^2 = 0$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin \frac{1}{x}$ and $h(x) = x^2$ in Squeeze theorem, we obtain $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

Application activity 8.2

1. Evaluate the following limits

   (a) $\lim_{x \to 2} 2x + 1$

   (b) $\lim_{a \to 1} a^2 - 1$

   (c) $\lim_{x \to -3} \frac{x^2 + x - 2}{x + 1}$

   (d) $\lim_{x \to -\infty} \frac{4x^2 + 3x - 2}{x^2 + x - 1}$

2. Find the limit of the following functions by squeeze theorem

   (a) $\lim_{x \to 0} x \cos 2x$

   (b) $\lim_{x \to +\infty} \frac{\sin x}{x}$; $\forall x \in [0, +\infty[$

   (c) $\lim_{x \to 0} \frac{\sin x}{x}$; $0 \le x < \frac{\pi}{2}$

8.3 Indeterminate forms

Suppose that $f(x) \to 0$ and $g(x)$ as $x \to a$. Then the limit of the quotient $\frac{f(x)}{g(x)}$ as $x \to a$ is said to give an indeterminate form, sometimes denoted by $\frac{0}{0}$. It may be that the limit of $\frac{f(x)}{g(x)}$ can be found by some methods such as factor method, rationalisation method, l'Hфpital's rule, etc...

Similarly, if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then the limit of $\frac{f(x)}{g(x)}$ gives an indeterminate form, denoted by $\frac{\infty}{\infty}$. Also, if $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$, then the limit of the product $f(x)g(x)$ gives an indeterminate form $0 \times \infty$.

Method of factors

Consider $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Putting $x = 3$ in the numerator and denominator the given fraction becomes an indeterminate form $\frac{0}{0}$, which is an undefined expression.

Thus, when $x = 3$ both numerator and denominator becomes zero and therefore $x - 3$ is a factor of both numerator and denominator. Now cancel the common factor $x - 3$.

$$\lim_{x \to 0} \frac{x^2 - 9}{x - 3} = \lim_{x \to 0} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 0} (x + 3) = 3 + 3 = 6$$

Hence, to evaluate $\lim_{x \to a} f(x)$, where $f(x)$ is an irrational function.

(1) Substitute x in a both numerator and denominator

(2) If the given fraction becomes an indeterminate form $\frac{0}{0}$, factorise numerator as well as denominator and cancel the non-zero common factor $x - a$.

(3) Then on putting $x = a$, we obtain $\lim\limits_{x \to a} f(x)$

Example 8.8

Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 3x + 2}$

Solution

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 3x + 2} = \frac{0}{0} \text{ (indeterminate form(I.F))}$$

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 3x + 2} = \lim_{x \to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x \to 2} \frac{(x+3)}{(x-1)} = \frac{2+3}{2-1} = \frac{5}{1} = 5$$

Method of substitution

For $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$, we follow these steps:

I Put $x = a + h$, where h is small and $h \neq 0$.

$h = x - a$ and as $x \to a$, $h \to 0$

II Simplify numerator and denominator and cancel h through $[h \neq 0]$

III As $h \to 0$, we get the limit of the simplified expression which is the required limits.

Example 8.9

Evaluate $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x^2 + 2x - 3}$

Solution

$$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 + 2x - 3} = \frac{1 - 4 + 3}{1 + 2 - 3} = \frac{0}{0} \text{ Indeterminate form (I.F)}$$

Put $x = 1 + h$ then $h = x - 1$ and as $x \to 1$ then $h \to 0$

$$\frac{x^2 - 4x + 3}{x^2 + 2x - 3} = \frac{(1+h)^2 - 4(1+h) + 3}{(1+h)^2 + 2(1+h) - 3} = \frac{1 + 2h + h^2 - 4 - 4h + 3}{1 + 2h + h^2 + 2 + 2h - 3} = \frac{h^2 - 2h}{h^2 + 4h} =$$

$$\frac{h(h-2)}{h(h+4)} = \frac{h-2}{h+4}$$

$$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 + 2x - 3} = \lim_{h \to 0} \frac{h-2}{h+4} = \frac{0-2}{0+4} = \frac{-2}{4} = -\frac{1}{2}$$

Method of rationalisation

In functions which involve square roots, rationalisation of either numerator or denominator and simplifications will facilitate the work.

Example 8.10

Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{0}{0} \text{ (indeterminate form)}$$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} = \frac{0}{0} \text{ (indeterminate form)}$$

$$\lim_{x\to 0} \frac{1}{(\sqrt{1+x}+1)} = \frac{1}{(\sqrt{1+0}+1)} = \frac{1}{2}$$

True value of limits

Computation of limits may results in indeterminate form (I.F) such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $\mathbf{0}.\infty$ that must be taken away. The following examples show how to find the true values:

Examples of the form $\frac{0}{0}$

Example 8.11

Compute $\lim_{x\to 2} \frac{x^2+x-6}{x^2-x-2}$.

Solution

For this limit, we find that both numerator and denominator are cancelled by taking $x = 2$ i.e $\lim_{x\to 2} \frac{x^2+x-6}{x^2-x-2} = \frac{0}{0}$ (indeterminate form)

The decomposition in factors followed by simplification gives the true value. i.e.

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-x-2} = \lim_{x\to 2} \frac{(x+3)(x-2)}{(x+1)(x-2)} = \lim_{x\to 2} \frac{x+3}{x+1} = \frac{5}{3}$$

Example 8.12

Compute $\lim\limits_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$.

Solution

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\frac{0}{0}\ \text{(indeterminate form)}$$

If there is an irrational expression, the true value is given by conjugating followed by a simplification. In our case, we find:

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}=\lim_{x\to 3}\frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$$

$$=\lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x+1}+2)}=\lim_{x\to 3}\frac{1}{(\sqrt{x+1}+2)}=\frac{1}{2+2}=\frac{1}{4}$$

Examples of the form $\frac{\infty}{\infty}$

Computing the following limits, allows us to conclude what is given after:

Example 8.13

Compute the following:

a) $$\lim_{x\to\infty}\frac{3x-4}{2x^2-3x+5}=\lim_{x\to\infty}\frac{x^2\left(\frac{3}{x}-\frac{4}{x^2}\right)}{x^2\left(2-\frac{3}{x}+\frac{5}{x^2}\right)}=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{4}{x^2}}{2-\frac{3}{x}+\frac{5}{x^2}}=\frac{0-0}{2-0+0}=\frac{0}{2}=0$$

b) $$\lim_{x\to\infty}\frac{2x^2-3}{5x^2+4x-5}=\lim_{x\to\infty}\frac{x^2\left(2-\frac{3}{x^2}\right)}{x^2\left(5+\frac{4}{x}-\frac{6}{x^2}\right)}=\lim_{x\to\infty}\frac{2-\frac{3}{x^2}}{5+\frac{4}{x}-\frac{6}{x^2}}=\frac{2-0}{5+0+0}=\frac{2}{5}$$

c) $$\lim_{x\to\infty}\frac{3x^3-2x+1}{-x^2+3x-5}=\lim_{x\to\infty}\frac{x^3\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{x^2\left(-1+\frac{3}{x}-\frac{5}{x^2}\right)}=\lim_{x\to\infty}\frac{x\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{-1+\frac{3}{x}-\frac{5}{x^2}}=\frac{3(+\infty)}{-1}=\frac{+\infty}{-1}=-\infty$$

d) $$\lim_{x\to-\infty}\frac{3x^3-2x+1}{-x^2+3x-5}=\lim_{x\to-\infty}\frac{x^3\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{x^2\left(-1+\frac{3}{x}-\frac{5}{x^2}\right)}=\lim_{x\to-\infty}\frac{x\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{-1+\frac{3}{x}-\frac{5}{x^2}}=\frac{3(-\infty)}{-1}=\frac{-\infty}{-1}=+\infty$$

Thus, we we can say that:

- If the degree of the numerator is smaller than the one of the denominator, the limit of the fraction as $x\to\infty$ is zero;
- If the degree of the numerator is equal to the one of the denominator, the limit of the fraction as $x\to\infty$ is equal to the quotient of the coefficient of the terms with the highest power;

- If the degree of the numerator is greater than the one of the denominator, the limit of the fraction as $x \to \infty$ gives infinite (the sign of infinite depends on the highest power and its coefficient).

Example 8.14

Compute $\lim\limits_{x\to\pm\infty} \frac{\sqrt{x^2+3x-5}}{3x+2}$

Solution

$$\lim_{x\to+\infty} \frac{\sqrt{x^2+3x-5}}{3x+2} = \lim_{x\to+\infty} \frac{\sqrt{x^2\left(1+\frac{3}{x}-\frac{5}{x^2}\right)}}{3x+2} = \lim_{x\to+\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{3x+2}$$

$$= \lim_{x\to+\infty} \frac{|x|\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{3x+2} = \lim_{x\to+\infty} \frac{x\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{(3+\frac{2}{x})} =$$

$$\lim_{x\to+\infty} \frac{\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{3+\frac{2}{x}} = \frac{1}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+3x-5}}{3x+2} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+\frac{3}{x}-\frac{5}{x^2}\right)}}{3x+2} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{3x+2}$$

$$= \lim_{x\to-\infty} \frac{|x|\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{3x+2} = \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{x(3+\frac{2}{x})} =$$

$$\lim_{x\to-\infty} \frac{-\sqrt{1+\frac{3}{x}-\frac{5}{x^2}}}{3+\frac{2}{x}} = -\frac{1}{3}$$

Examples of the form $\infty - \infty$ or/and the form $0.\infty$

Example 8.15

Compute $\lim\limits_{x\to\infty} (\sqrt{4x^2+3x+1}-2x)$

Solution

$$\lim_{x\to\infty}\left(\sqrt{4x^2+3x+1}-2x\right) = +\infty-\infty \text{ (indeterminate form)}$$

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+3x+1}-2x\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{4x^2+3x+1}-2x\right)\left(\sqrt{4x^2+3x+1}+2x\right)}{\sqrt{4x^2+3x+1}+2x}$$

$$=\lim_{x\to+\infty}\frac{4x^2+3x+1-4x^2}{\sqrt{4x^2+3x+1}+2x}=\lim_{x\to+\infty}\frac{3x+1}{\sqrt{4x^2+3x+1}+2x}$$

$$=\lim_{x\to+\infty}\frac{x\left(3+\frac{1}{x}\right)}{x\left(\sqrt{4+\frac{3}{x}+\frac{1}{x^2}}+2\right)}=\lim_{x\to+\infty}\frac{3+\frac{1}{x}}{\sqrt{4+\frac{3}{x}+\frac{1}{x^2}}+2}=\frac{3}{\sqrt{4}+2}=\frac{3}{4}$$

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x+1}-2x\right)=\lim_{x\to-\infty}-x\left(\sqrt{4+\frac{3}{x}+\frac{1}{x^2}}+2\right)$$

$$=+\infty\left(\sqrt{4}+2\right)=+\infty(4)=+\infty$$

Activity 8.3

Evaluate the limit: $\lim_{x\to\pm\infty}\sqrt{x^2+x+2}-\sqrt{x^2-x+3}$

Application activity 8.3

Find the following limits, if they exist.

1. $\lim_{x\to2}\frac{x^2-x-2}{x^2-3x+2}$
2. $\lim_{x\to3}\frac{\sqrt{25-x^2}-4}{x-3}$
3. $\lim_{x\to-\infty}\frac{4x+1}{x-6}$
4. $\lim_{x\to3}\frac{\sqrt{x}-\sqrt{3}}{x-3}$
5. $\lim_{x\to1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$
6. $\lim_{z\to0}\frac{\sqrt{2z+4}-2}{z}$
7. $\lim_{x\to+\infty}\left(\sqrt{x^2+x-2}-x\right)$
8. $\lim_{x\to0}\frac{x^2+2x}{x^3-3x}$
9. $\lim_{x\to\infty}\frac{10x^2-7x+5}{1+2x-5x^2}$
10. $\lim_{x\to0}\frac{\sqrt{3-x}-\sqrt{3}}{x}$
11. $\lim_{x\to2}\frac{x+3}{x-4}$
12. $\lim_{x\to1}\frac{(x+2)(x-1)}{x^2-6x+5}$
13. $\lim_{x\to-\infty}\frac{3x^3+5x^2-7}{5x-4x^3}$
14. $\lim_{m\to0}\frac{\sqrt{5+m}-\sqrt{5}}{m}$

15. $\lim_{x \to 3^+} \frac{x^2|x-3|}{x-3}$

16. $\lim_{x \to 2} \frac{x^2 + 3x - 10}{x - 2}$

17. $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{3x-1}$

18. $\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^3 - 3}$

19. $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 + x - 6}$

20. $\lim_{x \to -3^-} \frac{x-3}{x+3}$

21. $\lim_{x \to \infty} \frac{x - x^2}{x + x^2}$

22. $\lim_{x \to 0} \frac{x - x^2}{x + x^2}$

23. $\lim_{x \to -7} \frac{x^2 + 5x - 14}{x + 7}$

24. $\lim_{x \to \infty} 5$

25. $\lim_{x \to \infty} \frac{2x - 5}{x^2 + x + 1}$

26. $\lim_{x \to 4} \frac{x^2 - x - 12}{x - 4}$

27. $\lim_{x \to -2} \frac{x^2 + 7x + 10}{x + 2}$

28. $\lim_{x \to \infty} \frac{6x^3 + x^2 + 7x - 2}{5x^2 + 2x - 1}$

29. $\lim_{x \to \infty} \frac{x^2 + 3x}{1 - x^2 - x^3}$

30. $\lim_{x \to 3^-} \frac{10x^3 - x^2}{x - 3}$

31. $\lim_{x \to -2} (2x^2 - 6x + 5)^2$

32. $\lim_{x \to -5} \frac{x + 5}{x^2 + 4x - 5}$

33. $\lim_{x \to 0} \frac{(x+2)^2 - 4}{x}$

34. $\lim_{x \to \infty} \frac{3x^4 - 2}{\sqrt{x^8 + 3x + 4}}$

35. $\lim_{z \to \infty} - 2z$

36. $\lim_{x \to 2^-} \frac{(x-1)(x-2)}{x+1}$

37. $\lim_{x \to 2} \frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3}$

38. $\lim_{x \to -\infty} \frac{x - 2}{x^2 + 2x + 1}$

39. $\lim_{x \to 0} \frac{x}{|x|}$

40. $\lim_{x \to -4} \frac{3 - \sqrt{25 - x^2}}{x + 4}$

41. $\lim_{t \to 2} \frac{t^2 - t - 2}{t^2 - 3t + 2}$

42. $\lim_{h \to 3} \frac{h^2 + h + 1}{\sqrt{h + 6}}$

43. $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2-x}}{x}$

44. $\lim_{x \to \infty} \frac{3x^4 - 2x^2 + 1}{7x^4 + 6x^3 + x}$

45. $\lim_{x \to -\infty} \frac{\sqrt{x^2 + 2x + 2}}{x}$

46. $\lim_{x \to 2} \frac{x^2 - 2x + 3}{\sqrt{8x}}$

47. $\lim_{x \to 0} \left(\frac{2 - \sqrt{x}}{4 - x}\right)$

48. $\lim_{x \to 1^+} \frac{x^2 + 2x - 3}{x^2 - 2x + 1}$

49. $\lim_{x \to \infty} \frac{x^2 - x^{-2}}{4x^2 + 4x^{-2}}$

50. $\lim_{x \to 0} \frac{x^3 - 3x}{x^2 + x}$

51. $\lim_{t \to -\infty} \frac{x^2 + 6x + 5}{3x^2 + 4}$

52. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

53. $\lim_{x \to -2} \frac{(x + 2)(x^2 - 3x + 5)}{x^2 + 3x + 2}$

54. $\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 6x + 8}$

55. $\lim_{x \to 0} \frac{\sqrt{x + 9} - 3}{x}$

56. $\lim_{x \to \infty} \frac{3x^3 - 4x^2 + 6x - 5}{3 + x - x^2 - 2x^3}$

57. $\lim_{x \to 0} \frac{x^2 - 1}{x - 1}$

58. $\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x}$

59. $\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 - 4}$

60. $\lim_{x \to 0} \frac{2 - \sqrt{4 - x}}{3x}$

61. Given the function $f(x) = \begin{cases} \frac{x^2 - 1}{x^2 - x - 2} & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ \frac{\sqrt{x} - 2}{x - 4} & \text{if } x > 0 \end{cases}$

Evaluate the following limits:

a) $\lim_{x \to 0} f(x)$ b) $\lim_{x \to 4} f(x)$

c) $\lim_{x \to -1} f(x)$ d) $\lim_{x \to 2} f(x)$

62. Given the function $y = \frac{|x| - x}{x}$, find the following limits:

a) $\lim_{x \to 0^+} y$ b) $\lim_{x \to 0^-} y$

c) $\lim_{x \to 0} y$

8.4 Application of limits

Activity 8.4

Carry out research to find out the applications of limits. Discuss your findings with the rest of the class.

Continuity of a function at a point or interval

Let f be defined for all values of x near $x = x_0$ as well as $x = x_0$ (i.e., in a a δ neighbourhood of x_0). The function f is called continuous at $x = x_o$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Note that this implies the following three conditions. The function f(x) is continuous at $x = x_0$ if:

1. $f(x_0)$ exists.
2. $\lim_{x \to x_0} f(x) = L$ must exist.
3. $\lim_{x \to x_0} f(x) = f(x_0)$ i.e. $L = f(x_0)$.

In summary, $\lim_{x \to x_0} f(x)$ is the value suggested for f at $x = x_0$ by the behaviour of in arbitrary small neighbourhoods of x_0. In fact this limit is the actual value, $f(x_0)$, of the function at x_0 , then f is continuous there.

Equivalently, if f is continuous at x_0 , we can write this in the suggestive form

$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x)$.

Example 8.16

State whether or not, the following functions are continuous at x = 2.

1. $f(x) = \begin{cases} x^2, & x \neq 2 \\ 0, & x = 2 \end{cases}$.
2. $g(x) = x^3$ for all ,

Solution

1. $f(2) = 0$

   $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4.$

   We see that $\lim_{x \to 2} f(x) \neq f(2)$ i.e. $4 \neq 0$. Thus the function is not continuous at x = 2 .
2. $f(2) = 2^3 = 8$

   $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 8$.

   We see that $\lim_{x \to x_0} f(x) = f(x_0)$ i.e. 8 = f(2). Thus, f(x) is continuous at x = 2 .

Points where f fails to be continuous are called discontinuities of f and f is said to be discontinuous at these points.

Continuity over an interval

The function is said to be continuous over interval]a, b[if and only if f(x) is continuous at any point of the interval]a, b[. i.e.

f is continuous for all $x_0 \in]a, b[$. f is continuous at the right side of a. f is continuous at the left side of b .

Example 8.17

Determine whether or not the function $f(x) = \frac{1}{x^2 - 1}$ is continuous over the interval] – 2, 2[.

Solution

$Df = \mathbb{R} \setminus \{-1,1\}$

f is continuous at right side of –2. $f(-2) \in \mathbb{R}$, $2 \in Df$

is continuous at left side of 2.

At $x_0 = -1$ and $x_0 = 1$, the function is discontinuous and those point are inside the interval] –2, 2[. Thus, the function is discontinuous over] –2, 2[.

Properties

1. The sum of two continuous functions is a continuous function.
2. The quotient of two continuous functions is a continuous function where the denominator is not zero.

Asymptotes

Activity 8.5

What is an asymptote? Carry out research and find out the meaning. Also, find out the types of asymptotes.

An asymptote is a **line** that a curve approaches, as it heads towards infinity:

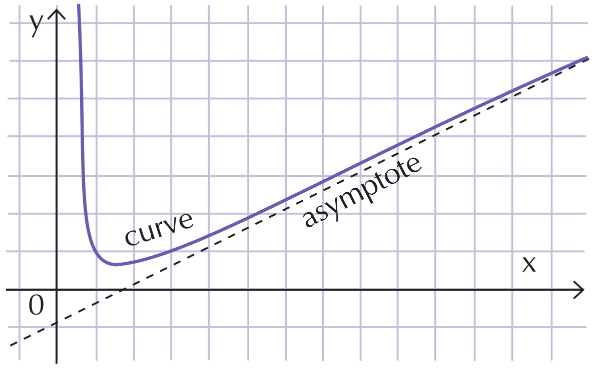


Fig. 8.5

Types

There are three types of asymptotes: horizontal, vertical and oblique:

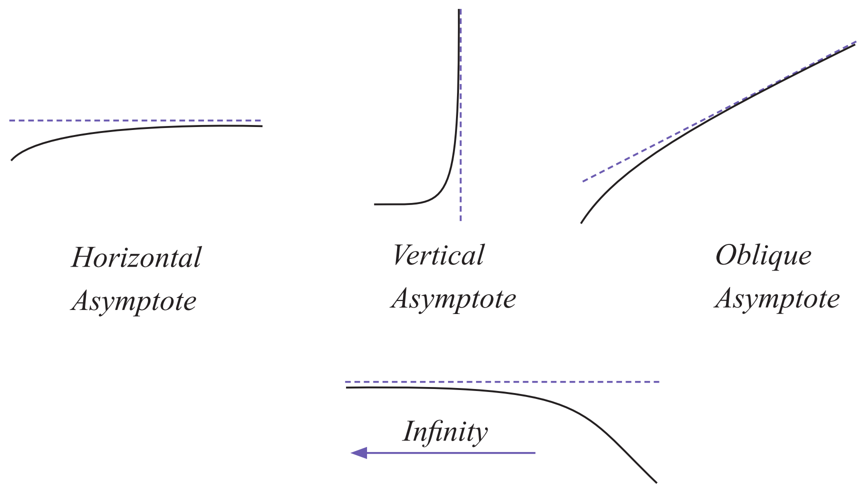


Fig. 8.6

An asymptote can be in a negative direction, the curve can approach from any side (such as from above or below for a horizontal asymptote), or may actually cross over (possibly many times), and even move away and back again.

The important point is that:

The **distance** between the curve and the asymptote **tends to zero** as they head to infinity (or – infinity)

Horizontal asymptotes

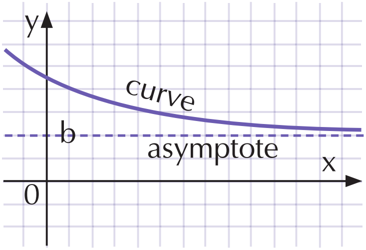


Fig. 8.7

It is a horizontal asymptote when:
as x goes to infinity (or –infinity) the curve approaches some constant value **b**

$$\lim_{x \to \pm\infty} f(x) = b$$

Example 8.18

Find the equation of the horizontal asymptote to the curve $f(x) = \frac{x^2 - 1}{x^2 + 2}$

Solution

Since $\lim_{x \to \pm\infty} \frac{x^2 - 1}{x^2 + 2} = 1$, the curve representing $f(x) = \frac{x^2 - 1}{x^2 + 2}$ has the horizontal asymptote (H.A) ≡ $y = 1$.

Vertical asymptotes

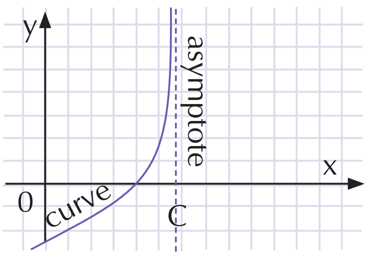


Fig. 8.8

It is a vertical asymptote when:
as x approaches some constant value **c** (from the left or right) then the curve goes towards infinity (or – infinity).

Example 8.19

Find the equation of the vertical asymptote to the curve $f(x) = \frac{x^3 + x + 1}{x - 2}$

Solution

Since for example $\lim_{x \overset{<}{\to} 2} f(x) = \frac{8 + 2 + 1}{0^-} = \frac{11}{0^-} = -\infty$, the function $f(x) = \frac{x^3 + x + 1}{x - 2}$ has a vertical asymptote **V.A ≡ x = 2.**

Oblique asymptotes

It is an oblique asymptote when:

as x goes to infinity (or –infinity) then the curve goes towards a line **y = mx + b**

(note: m is not zero as that is a horizontal asymptote).

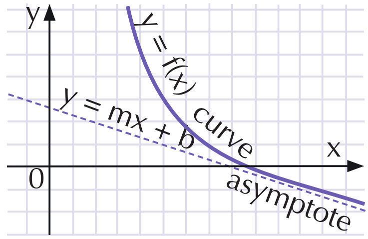


Fig. 8.9

The characteristics of the three kinds of asymptotes:
vertical asymptote**, horizontal** asymptote and o**blique** asymptote are:

1. A line whose equation is **H.A** ≡ **y = b** is the **horizontal asymptote** to the curve y = f(x) if either

   $\lim\limits_{x\to\infty} f(x) = b$ or $\lim\limits_{x\to-\infty} f(x) = b$

2. A line whose equation is V.A ≡ x = a is a vertical asymptote to the curve y = f(x) if at least one of the following statements is true:

   $\lim\limits_{x\to a} f(x) = \infty \qquad \lim\limits_{\substack{x\to a\\ <}} f(x) = \infty \qquad \lim\limits_{\substack{x\to a\\ >}} f(x) = \infty$

   $\lim\limits_{x\to a} f(x) = -\infty \qquad \lim\limits_{\substack{x\to a\\ <}} f(x) = -\infty \qquad \lim\limits_{\substack{x\to a\\ >}} f(x) = -\infty$

3. The line of equation **O.A** ≡ **y = ax + b** is the **oblique asymptote** to the graph f iff .

   $\lim\limits_{x\to\pm\infty} [f(x) - (ax + b)] = 0$

   The real numbers a and b are obtained by

   $a = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x}$ and $b = \lim\limits_{x\to\pm\infty} [f(x) - ax]$.

Example 8.20

Find the equation of the oblique asymptote to the curve representing $f(x) = \frac{x^2 + 3x - 1}{2x + 1}$

Solution

This asymptote is given by y = ax + b where

$a = \lim\limits_{x\to\infty} \frac{f(x)}{x} = a = \lim\limits_{x\to\infty} \frac{x^2 + 3x - 1}{x(2x + 1)} = a = \lim\limits_{x\to\infty} \frac{x^2+3x - 1}{2x^2 + x} = \frac{1}{2}$ and

$$b = \lim_{x\to\infty} [f(x) - ax] = \lim_{x\to\infty} \left[\frac{x^2 + 3x - 1}{2x + 1} - \frac{1}{2}x\right] = \lim_{x\to\infty} \left[\frac{2x^2 + 6x - 2 - 2x^2 - x}{4x + 2}\right]$$

$$= \lim_{x\to\infty} \frac{5x - 2}{4x + 2} = \frac{5}{4}$$

Hence the oblique asymptote is O.A ≡ $y = \frac{1}{2}x + \frac{5}{4}$

Note: For the rational functions;

- o There is the oblique asymptote if the degree of the numerator is one greater than the degree of the denominator. An alternative way to find the equation of the oblique asymptote is to use a long division. There the equation is simply the quotient.
- o The vertical asymptote, are found to be the values that make the denominator zero
- o For the horizontal asymptote, if the degree of the numerator is less than the degree of the denominator, then H.A ≡ $y = 0$. If the degree of the numerator is or equal to the degree of the denominator, then H.A ≡ $y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
- o The horizontal asymptote is the special case of the oblique asymptote.

Example 8.21

Find all the equations of the asymptotes of the following functions:

(a) $y = \frac{x + 3}{(x + 2)(x - 1)}$ (b) $y = \frac{3x^2}{x^2 + x + 1}$

c) $y = \frac{x^3}{x^2 + x + 1}$

Solution

(a) $y = \frac{x + 3}{(x + 2)(x - 1)}$

There are two vertical asymptotes V.A ≡ $x = -2$ and V.A ≡ $x = 1$

The horizontal asymptote has equation H. A ≡ $y = 0$

There is no oblique asymptote

(b) $y = \frac{3x^2}{x^2 + x + 1}$

No vertical asymptote. Horizontal asymptote is H.A ≡ $y = 3$. No oblique asymptote.

(c) $y = \frac{x^3}{x^2 + x + 1}$

No vertical asymptote. No Horizontal asymptote.

For the oblique asymptote, let us use the long division

$$\frac{x^3}{x^2 + x + 1} = \frac{x(x^3 + x + 1) - x^2 - x}{x^2 + x + 1} = \frac{x(x^2 + x + 1 - x - 1)}{x^2 + x + 1} = \frac{x(x^2 + x + 1) - (x^2 + x + 1) + 1}{x^2 + x + 1}$$

$$= x - 1 + \frac{1}{x^2 + x + 1}$$

Thus, the oblique asymptote has equation O.A $\equiv y = x - 1$.

Graphing asymptotes

Example 8.22

Find the asymptotes of $\frac{(x^2 - 3x)}{(2x - 2)}$ and sketch the graph.

Solution

There is one vertical asymptote V.A $\equiv x = 1$.

There is no horizontal asymptote.

The oblique asymptote is found by long division $\frac{(x^2 - 3x)}{(2x - 2)} = \frac{1}{2}x - 1 - \frac{2}{2x - 2}$.

So O.A $\equiv y = \frac{1}{2}x - 1$ and the sketch graph is:

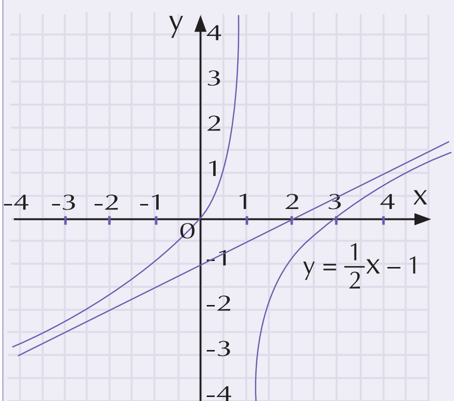


Fig. 8.10

To graph a rational function, you find the asymptotes and the intercepts, plot a few points, and then sketch in the graph.

Example 8.23

- Graph the following and find the vertical asymptote, if any, for this rational function.

$$y = \frac{2x + 5}{x - 1}$$

Solution

First find the vertical asymptotes, if any, for this rational function. Since we cannot graph where the function does not exist, and since the function won't exist where there would be a zero in the denominator, we set the denominator equal to zero to find any unwanted points:

$x - 1 = 0$

$x = 1$

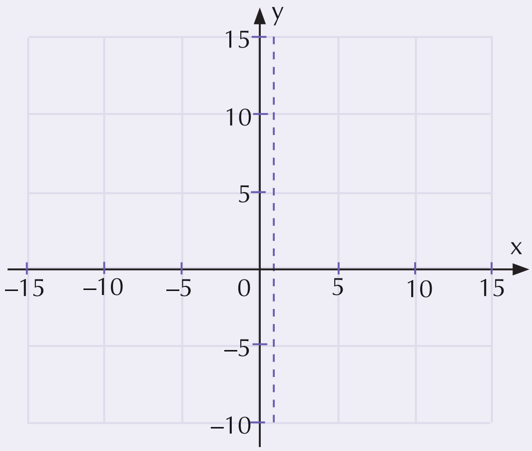


Fig. 8.11

We cannot have $x = 1$, and therefore we have a vertical asymptote there.

This is shown as a dotted line on the graph:

Next we find the horizontal or slant asymptote. Since the numerator and denominator have the same degree (they are both linear), the asymptote will be horizontal, not slant, and the horizontal asymptote will be the result of dividing the leading coefficients:

$y = \frac{2}{1} = 2$

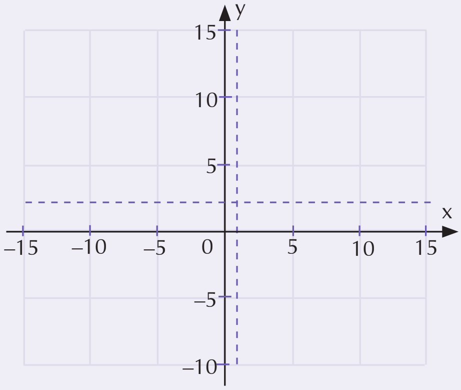


Fig. 8.12

We also show this by a dotted line:

Next, we find any *x*- or *y*-intercepts.

$x = 0$: $y = \frac{(0 + 5)}{(0 - 1)} = \frac{5}{-1} = -5$

$y = 0$: $0 = \frac{(2x + 5)}{(x - 1)}$

$0 = 2x + 5$

$-5 = 2x$

$-2.5 = x$

Then the intercepts are at (0, –5) and (–2.5, 0). We sketch these in:

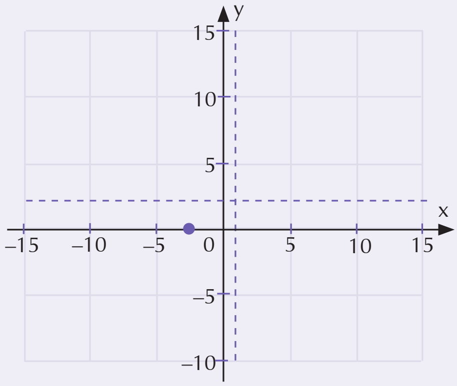


Fig. 8.13

Now we pick a few more *x*-values, compute the corresponding *y*-values, and plot a few more points.

x	$y = \frac{(2x + 5)}{(x - 1)}$
–6	$\frac{(2(-6) + 5)}{((-6) - 1)} = \frac{(-12 + 5)}{(-7)} = \frac{(-7)}{(-7)} = 1$
–1	$\frac{(2(-1) + 5)}{((-1) - 1)} = \frac{(-2 + 5)}{(-2)} = \frac{(3)}{(-2)} = -1.5$
2	$\frac{(2(2) + 5)}{((2) - 1)} = \frac{(4 + 5)}{(1)} = \frac{(9)}{(1)} = 9$
3	$\frac{(2(3) + 5)}{((3) - 1)} = \frac{(6 + 5)}{(2)} = \frac{(11)}{(2)} = 5.5$
6	$\frac{(2(6) + 5)}{((6) - 1)} = \frac{(12 + 5)}{(5)} = \frac{(17)}{(5)} = 3.4$
8	$\frac{(2(8) + 5)}{((8) - 1)} = \frac{(16 + 5)}{(7)} = \frac{(21)}{(7)} = 3$
15	$\frac{(2(15) + 5)}{((15) - 1)} = \frac{(30 + 5)}{(14)} = \frac{(35)}{(14)} = 2.5$

Now we plot these points:

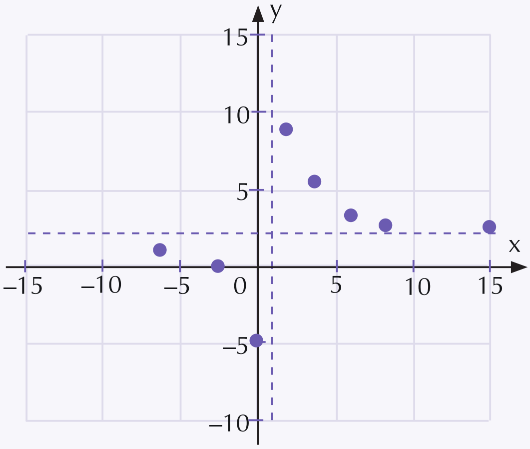


Fig. 8.14

And connect the dots:

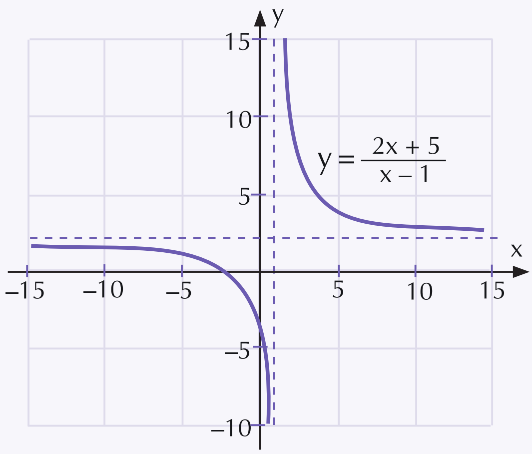


Fig. 8.15

Application activity 8.4

Graph the following and find their asymptotes.

1. $y = \frac{x + 2}{x^2 + 1}$
2. $y = \frac{x^3 - 8}{x^2 + 5x + 6}$
3. $y = \frac{x^2 - x - 2}{x - 2}$

Summary

1. A **neighbourhood** of a real number is any interval that contains a real number a and some point below and above it.
2. If x is taking values sufficiently close to and greater than a, then we say that x tends to a from above and the limiting value is then what we call the **right-sided limit**. It is written as $\lim\limits_{x\to a^+} f(x) = \lim\limits_{\substack{x\to a\\ >}} f(x)$
3. If x is taking values sufficiently close to and less than a, then we say that x tends to a from below and the limiting value is then what we call the **left-sided limit**. It is written as $\lim\limits_{x\to a^-} f(x) = \lim\limits_{\substack{x\to a\\ <}} f(x)$
4. If the f(x) tends closer to a value L as x approaches the value a from either side, then L is the limit of f(x) as x approaches . We use the following notation:

   $\lim\limits_{x\to a} f(x) = L$.
5. If f is a **polynomial** or a rational function and a is in the domain of f, then

   $\lim\limits_{x\to a} f(x) = f(a)$.
6. If a function can be **squeezed (sandwiched)** between two other functions, each of which approaches the same limit b as $x \to a$, then the squeezed function also approaches the same limit as $x \to a$.
7. A function is said to be **continuous** over interval]a, b[if and only if f(x) is continuous at any point of the interval]a, b[.
8. The sum of two continuous functions is a continuous function.
9. The quotient of two continuous functions is a continuous function where the denominator is not zero.
10. A line L is an **asymptote** to a curve if the distance from a point P of the curve to the line L tends to zero as P tends to infinity along some unbounded part of the curve. We have three kinds of asymptotes: vertical asymptote, horizontal asymptote and oblique asymptote.

Topic area: Analysis

Sub-topic area: Limits, differentiation and integration

Unit 9

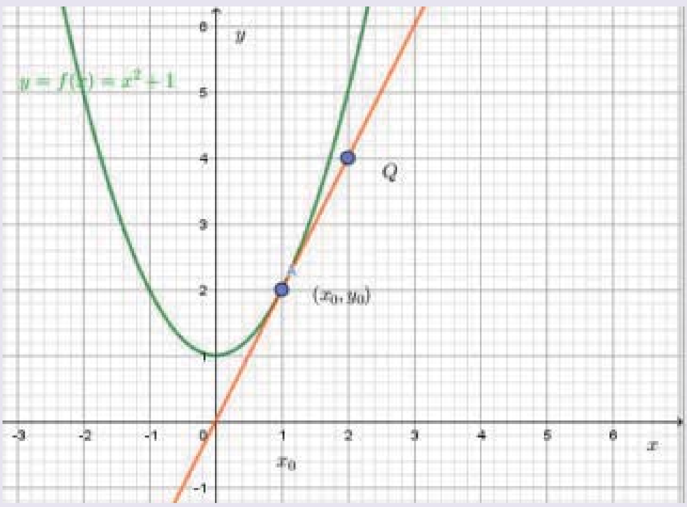
Differentiation of polynomials, rational and irrational functions, and their applications

Key unit competence

Use the gradient of a straight line as a measure of rate of change and apply this to line tangents and normals to curves in various contexts. Use the concepts of differentiation to solve and interpret related rates and optimization problems in various contexts.

9.0 Introductory activity

1. Consider the function $f(x)=x^2+1$ illustrated on the following graph;



It is defined that the slope m_P of the tangent of the curve of $f(x)$ at a point $P(x_0, y_0)$ is obtained by $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$,

a) Determine the slope of $f(x)$ at the point for which $x_0=1$.

b) Deduce the value of the function $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2+1$ and compare the slope m_P and $f'(x_0)$ for $x_0=1$.

2) Go in library or computer lab, do research and make a short presentation on the meaning of the derivative of a function.

9.1 Concepts of derivative of a function

Definition

Mental task

You have learnt about gradients. What is the gradient of a line? How is it obtained?

For a non-linear function with equation y = f(x), slopes of tangents at various points continually change.

The gradient of a curve at a point depends on the position of the point on the curve and is defined to be the gradient of the tangent to the curve at that point.

In Figure 9.1, P(x, f(x)) is any point on the graph of y = f(x) and Q is a neighbouring point (x +h, f(x + h)).

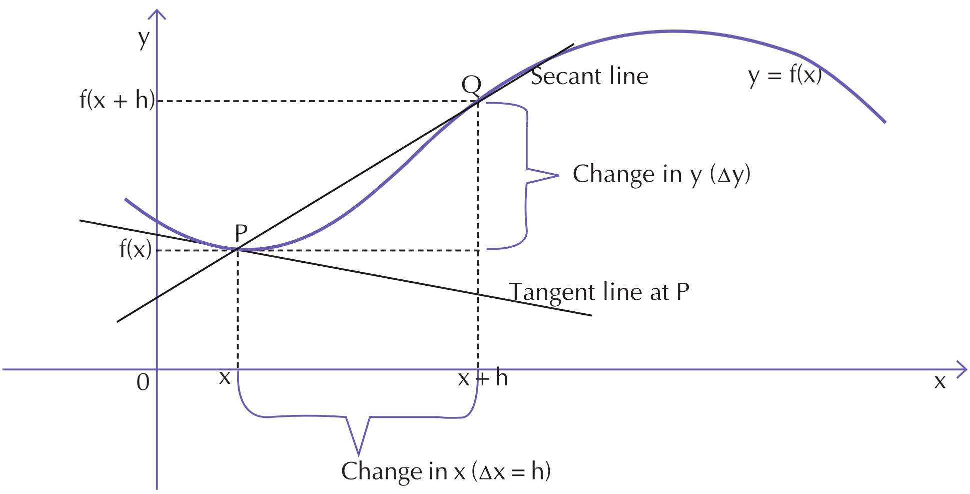


Fig. 9.1

As Q approaches P along the curve, the gradient of the secant PQ approaches the gradient of the tangent at P. The gradient of the tangent at P is thus defined to be the limit of the gradient of the secant PQ as Q approaches P along the curve. i.e., as $h \to 0$.

Now the gradient of PQ is $\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$

Thus, we define the gradient of the tangent at P and hence the gradient of the curve at P to be

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Example 9.1

Find the gradient of the curve $f(x) = x^2$ at the point P(1, 1).

Solution

Then the gradient of the secant PQ is

$$\frac{f(x + h) - f(x)}{(x + h) - x} = \frac{(x + h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$$

Therefore the gradient of the curve at P is $\lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0}(2x + h) = 2x$

At the point (1, 1), the gradient is 2(1) = 2.

Derivative of a function

Activity 9.1

Carry out research to find out the meaning of the term derivative in mathematics. Present your findings to the rest of the class.

A derivative is all about slope.

$$\textbf{Slope} = \frac{\text{Change in y}}{\text{Change in x}}$$

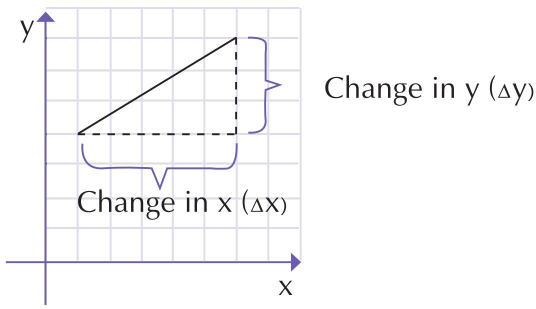


Fig. 9.2

We can find an **average** slope between two points.

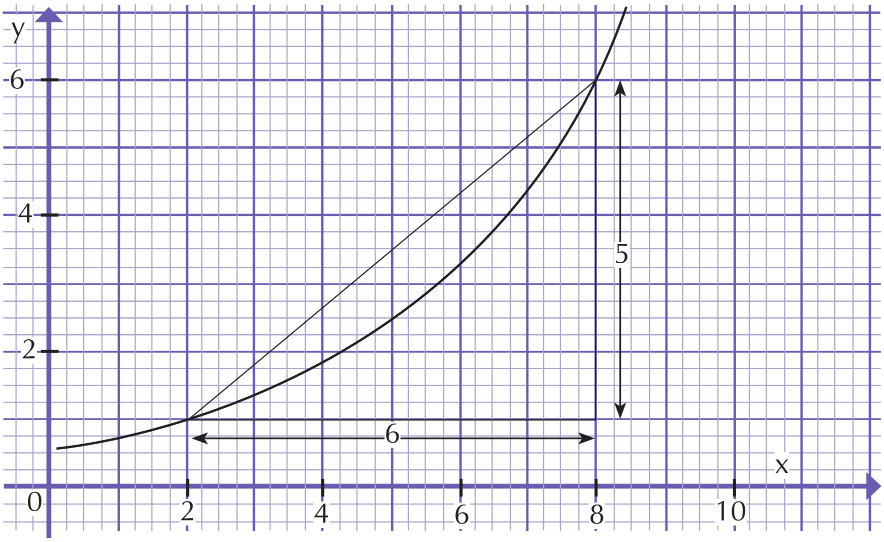


Fig. 9.3

Average slope is $\frac{5}{6}$

How do we get the slope (gradient) at a point?

The derivative of a function y = f(x) at the point (x, f(x)) equals the slope of the tangent line to the graph at that point.

Let us illustrate this concept graphically.

Let f be a real-valued function and P (x, f(x)) be a point on the graph of this function. Let there be another point Q in the neighbourhood of P.

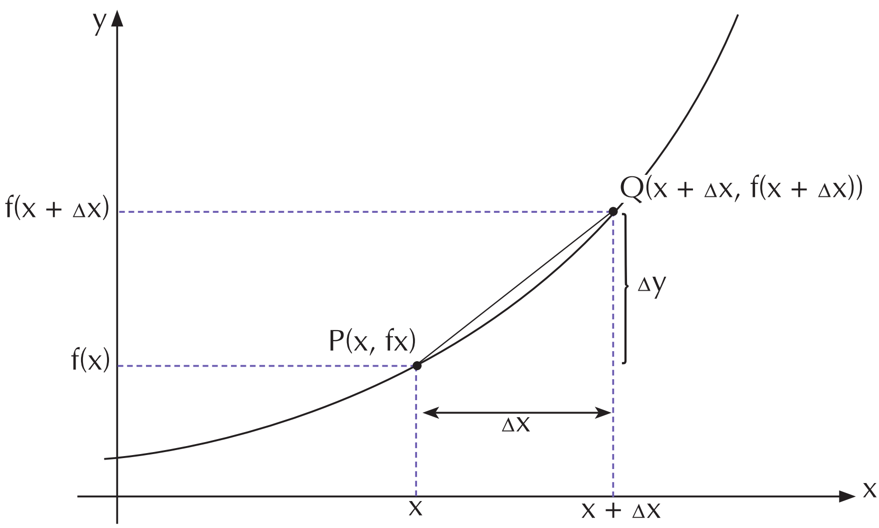


Fig 9.4

It is assumed that this point Q is extremely close to P, thus the coordinates of Q are $(x + \Delta x, f(x + \Delta x))$; where, "Δx" is a value which is very very small and provides an appropiate approximation to the slope of tangent line.

The straight line PQ has a gradient of $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$.

We now consider what happens to the slope of PQ as Δx gets smaller and smaller. i.e. Q gets nearer to P.

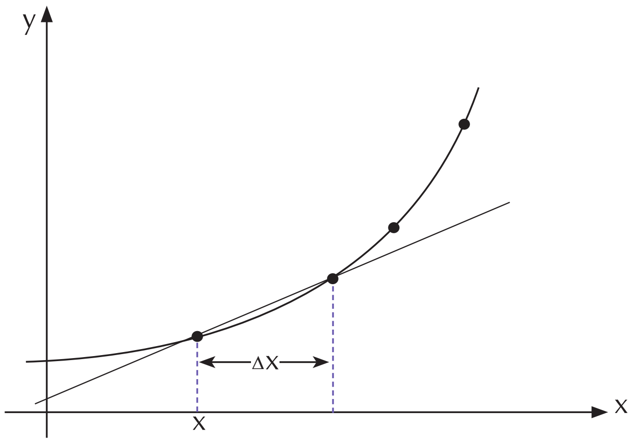


Fig 9.5

As the value of Δx gets smaller, the two points get closer and the slope of PQ approaches that of the tangent line to the curve at P. As this happens the gradient of PQ will get closer to the slope of the tangent at P.

If we take this to the limit, as Δx approaches 0, we will find the slope of the tangent at P and hence the gradient of the curve at P.

Gradient at P = $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Thus, $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ is the slope of the tangent at the point (x, f(x) and is called the **derivative of the function** f(x).

The derivative of the function with respect to x is the function and is defined as, $\mathbf{f'(x)} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ or $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

Let us see an example.

Take the function $f(x) = x^2$.

We know $\mathbf{f(x) = x^2}$, and can calculate $\mathbf{f(x + \Delta x)}$:

Start with: $\mathbf{f(x + \Delta x) = (x + \Delta x)^2}$

Expand $(x + \Delta x)^2$: $\mathbf{f(x + \Delta x) = x^2 + 2x\,\Delta x + (\Delta x)^2}$

The derivative of a function is given by: $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in **f(x+Δx)** and **f(x)**: $\frac{x^2 + 2x\,\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

Simplify (x^2 and $-x^2$ cancel): $\frac{2x\,\Delta x + (\Delta x)^2}{\Delta x}$

Simplify more (divide through by Δx): $= 2x + \Delta x$

And then **as Δx tends towards 0** we get: $= 2x$

Result: the derivative of $\mathbf{x^2}$ is **2x**

Note:

The slope (gradient) of the tangent to a curve of f(x) is defined as the slope of the curve f(x), and is the instantaneous rate of change in y with respect to x.

Finding the slope using the limit method is said to be using first principles.

A chord (secant) of curve is a straight line segment which joins any two points on the curve.

A tangent is straight line which touches curve at point.

The derivative of a function, also known as slope of a function, or derived function or simply the derivative, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Activity 9.2

Work out the following in groups.

The graph below shows part of the graph of the function $y = 0.1x^2 - 5$. Find the average slope between the points (2, – 4.6) and (10, 5).

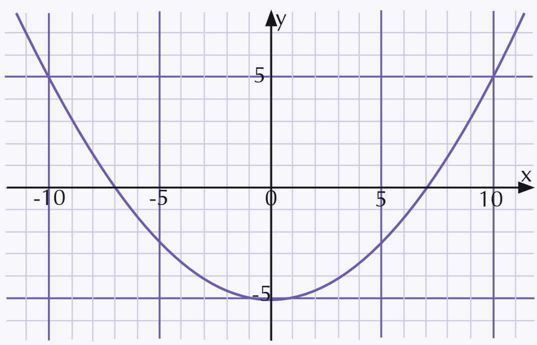


Fig. 9.6

Differentiation from first principles

Finding the slope using the limit method is said to be using first principles, that is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 9.2

Find, from the first principles, the derivative functions of

a) $f(x) = x^2$ b) $f(x) = \frac{1}{x}$ c) $f(x) = \sqrt{x}$

Solution

a) $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x$$

b) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left(\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \right) = \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} =$$

$$\lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x(x+0)} = -\frac{1}{x^2}$$

c) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} =$$

$$= \frac{1}{2\sqrt{x}}$$

Application activity 9.1

1. Determine from first principles the derivative of the following functions:

   (a) 2 (b) $x + x^3$ (c) $x^3 + 2x + 3$

   (d) $x^4 - \frac{1}{3}x$

2. If $f(x) = 4x + 2x^2$, find:

   (a) $f'(x)$ using the definition of derivative

   (b) $f'(2)$

   (c) $f'(-2)$

3. For each of the following, find, from first principles, the derivative:

   a) $\frac{1}{x+2}$ b) $\frac{1}{2x-1}$

   c) $\frac{1}{x^2}$ d) $\frac{1}{x^3}$

4. Find, from first principles, the derivative of the following functions:

   a) $\sqrt{x+2}$ b) $\frac{1}{\sqrt{x}}$

   c) $\sqrt{2x+1}$.

Alternative method

The derivative of a function f(x) at x = a is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Alternative formula for finding $f'(a)$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Thus, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ is the slope of the tangent at x and is called the derivative at x = a .

The slope of the tangent at the point x = a is defined as the slope of the curve at the point where x= a, and is the instantaneous rate of change in y with respect to x at that point.

Example 9.3

Find from first principles, the slope of the tangent to the following functions at a given value of x:

a) $f(x) = 2x^2 + 3$ at $x = 2$

b) $f(x) = \frac{2x - 1}{x + 3}$ at $x = -1$

c) $y = \sqrt{x}$ at $x = 9$

Solution

a) $f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ where $f(2) = 2(2)^2 + 3 = 11$

$f'(2) = \lim_{x \to 2} \frac{2x^2 + 3 - 11}{x - 2} = \lim_{x \to 2} \frac{2x^2 - 8}{x - 2} = \lim_{x \to 2} \frac{2(x^2 - 4)}{x - 2} =$

$\lim_{x \to 2} \frac{2(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} 2(x + 2) = 2(2 + 2) = 2 \times 4 = 8$

b) $f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \left(\frac{\frac{2x - 1}{x + 3} + \frac{3}{2}}{x + 1} \right) = \lim_{x \to -1} \left(\frac{\frac{2x - 1}{x + 3} + \frac{3}{2}}{x + 1} \right)$

$= \lim_{x \to -1} \frac{2(2x - 1) + 3(x + 3)}{2(x + 1)(x + 3)} = \lim_{x \to -1} \frac{4x - 2 + 3x + 9}{2(x + 1)(x + 3)} = \lim_{x \to -1} \frac{7x + 7}{2(x + 1)(x + 3)}$

$= \lim_{x \to -1} \frac{7(x + 1)}{2(x + 1)(x + 3)} = \lim_{x \to -1} \frac{7}{2(x + 3)} = \frac{7}{2(2)} = \frac{7}{4}$

c) $f'(9) = \lim_{x \to 9} \frac{f(x) - f(9)}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x} - 3}{(\sqrt{x} + 3)(\sqrt{x} - 3)} =$

$\lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$

Application activity 9.2

1. Find, from first principles, the slope of the tangent to the following functions at the given value of x:
   a) $f(x) = x^2 + 2x + 4$ at $x = 1$
   b) $f(x) = 1 - x^2$ at $x = 2$
   c) $f(x) = 7x - x^2$ at $x = 1$
   d) $f(x) = 2x^2 + 5x$ at $x = -1$
   e) $f(x) = 5 - 2x^2$ at $x = 3$
   f) $f(x) = 3x + 5$ at $x = -2$
2. Find, from the first principles, the derivative of the following functions at an indicated value of x:
   a) $f(x) = \frac{1}{x}$ at $x = 1$
   b) $f(x) = \frac{4}{x}$ at $x = 2$
   c) $f(x) = \frac{3}{x}$ at $x = -2$
   d) $f(x) = 2x - \frac{1}{x}$ at $x = 1$
   e) $f(x) = \frac{1}{x^2}$ at $x = 4$
   f) $f(x) = \frac{4x}{x - 3}$ at $x = 2$

g) $f(x) = \frac{4x+1}{x-2}$ at $x = 1$ h) $f(x) = \frac{3x}{x^2+1}$ at $x = -4$

3. Find, from first principles, the instantaneous rate of change in the following functions at the given point:

a) $\sqrt{x}$ at $x = 1$ b) $\sqrt{x}$ at $x = \frac{1}{4}$

c) $\frac{2}{\sqrt{x}}$ at $x = 9$ d) $\sqrt{x+1}$ at $x = 3$

Higher order derivatives

If f is a function which is differentiable on its domain, then f' is a derivative function.

If, in addition, f' is differentiable on its domain, then the derivative of f' exists and is denoted by f''; it is the function given by $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ and is called the **second derivative** of f.

If, in addition, f'' is differentiable on its domain, then the derivative of f'' is denoted by f''' and is called the **third derivative** of f .

In general, the **n^{th} derivative of f**, where n is a positive integer, denoted by $f^{(n)}$, is defined to be the derivative of the $(n-1)^{th}$ derivative of f. For $n > 1$, $f^{(n)}$ is called a higher-order derivative of f. If we differentiate the function y n times in successive differentiation, the resulting function is called the **n^{th} derivative of y** and is denoted as $\frac{d^ny}{dx^n}$ or $y^{(n)}$ or $f^{(n)}(x)$. The process of finding the successive derivatives of a function is known as successive differentiation.

Thus, if $y = f(x)$, the successive differential of f(x) are $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$

Note: We have different notations for second order derivative of f: y'', f'', $\frac{d^2y}{dx^2}$, $f''(x)$, $\frac{d^2}{dx^2}f(x)$, … . The 0^{th} derivative of f means f. For $n = 1$, the first derivative of f is simply the derivative of f and we already saw different notations of it.

Example 9.4

Find, from the first principles, the first, second and third derivatives of the following functions:

a) $f(x) = 5x^3 + 2$ b) $f(x) = \frac{x+3}{x}$

Solution

a) $f(x) = 5x^3 + 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5(x+h)^3 + 2 - (5x^3 + 2)}{h}$$

$$= \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) + 2 - 5x^3 - 2}{h} = \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3 - x^3)}{h}$$

$$= 5\lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 5\lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= 5\lim_{h \to 0} \frac{(3x^2 + 3xh + h^2)}{1} = \frac{(3x^2 + 0 + 0)}{1} = 5(3x^2) = 15x^2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h} = \lim_{h \to 0} \frac{15(x + h)^2 - 15x^2}{h} = \lim_{h \to 0} \frac{15(x^2 + 2xh + h^2) - 15x^2}{h}$$

$$= \lim_{h \to 0} \frac{15(x^2 + 2xh + h^2 - x^2)}{h} = 15\lim_{h \to 0} \frac{(2xh + h^2)}{h} = 15\lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= 15\lim_{h \to 0} \frac{(2x + h)}{1} = 15\left(\frac{(2x + 0)}{1}\right) = 15(2x) = 30x$$

$$f'''(x) = \lim_{h \to 0} \frac{f''(x + h) - f''(x)}{h} = \lim_{h \to 0} \frac{30(x + h) - 30x}{h} = \lim_{h \to 0} \frac{30x + 30h - 30x}{h}$$

$$= \lim_{h \to 0} \frac{30h}{h} = \lim_{h \to 0} 30 = 30$$

b) $f(x) = \frac{x + 3}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \left(\frac{\frac{x + h + 3}{x + h} - \frac{x + 3}{x}}{h}\right) = \lim_{h \to 0} \left(\frac{\frac{x^2 + xh + 3x - (x + 3)(x + h)}{x(x + h)}}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{\frac{x^2 + xh + 3x - x^2 - xh - 3x - 3h)}{x(x + h)}}{h}\right) = \lim_{h \to 0} \left(\frac{\frac{-3h}{x(x + h)}}{h}\right)$$

$$= \lim_{h \to 0} \frac{-3h}{xh(x + h)} = \lim_{h \to 0} \frac{-3}{x(x + h)} = -\frac{3}{x(x + 0)} = -\frac{3}{x^2}$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h} = \lim_{h \to 0} \frac{\left(-\frac{3}{(x + h)^2}\right) - \left(-\frac{3}{x^2}\right)}{h} = \lim_{h \to 0} \frac{\frac{-3}{x^2 + 2xh + h^2} + \frac{3}{x^2}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\frac{-3x^2 + 3x + 6xh + 3h^2}{x^2(x^2 + 2xh + h^2)}}{h}\right) = \lim_{h \to 0} \frac{+6xh + 3h^2}{x^2h(x^2 + 2xh + h^2)} = \lim_{h \to 0} \frac{+3(2x + h)}{x^2h(x^2 + 2xh + h^2)}$$

$$= \frac{+3(2x + 0)}{x^2(x^2 + 0 + 0)} = \frac{6x}{x^4} = \frac{6}{x^3}$$

$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{\frac{6}{(x+h)^3} - \frac{6}{x^3}}{h} = \lim_{h\to 0} \frac{\frac{6}{x^3+3x^2h+3xh^2+h^3} - \frac{6}{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{6x^3 - 6x^3 - 18x^2h - 18xh^2 - 6h^3}{x^3(x^3+3x^2h+3xh^2+h^3)}}{h} = \lim_{h\to 0} \frac{-6x(3x^2+3xh+h^2)}{hx^3(x^3+3x^2h+3xh^2+h^3)}$$

$$= \lim_{h\to 0} \frac{-6(3x^2+3xh+h^2)}{x^3(x^3+3x^2h+3xh^2+h^3)} = \frac{-6(3x^2+0+0)}{x^3(x^3+0+0+0)} = -\frac{18x^2}{x^6} = -\frac{18}{x^4}$$

Application activity 9.3

Find, from the first principles, the first and the second derivatives of the following functions:

a) $f(x) = x^3 + 2x + 3$

b) $f(x) = \frac{3x+1}{x-2}$

c) $f(x) = \frac{2-x}{x}$

d) $f(x) = \frac{x^2+x+1}{x+2}$

e) $f(x) = 2x^2 + \frac{1}{x} - \sqrt{x}$

f) $f(x) = \frac{3x^2+x-4}{x^2+2x-1}$

9.2 Rules of differentiation

Differentiation is the process of finding the derivative function. If we are given a function $f(x)$ then $f'(x)$ represents the derivative function. However, if we are given y in terms of x then y' or $\frac{dy}{dx}$ are usually used to represent the derivative function.

Note: $\frac{dy}{dx}$ reads 'dee y dee x', or 'the derivative of y with respect to x'. $\frac{dy}{dx}$ is not a fraction.

We write **dx** instead of **"Δx heads towards 0"**, so "the derivative of" is commonly written as $\frac{d}{dx}$, so $\frac{d}{dx}(x^2) = 2x$.

"The derivative of $\mathbf{x^2}$ equals $\mathbf{2x}$.

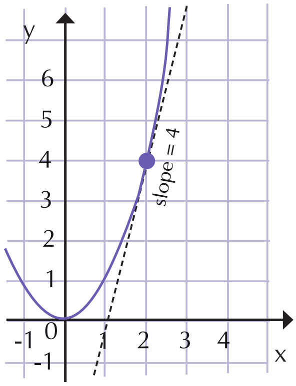


Fig. 9.7

What does $\frac{d}{dx} x^2 = 2x$ mean?

It means that, for the function x^2, the slope or "rate of change" at any point is **2x.**

So when **x = 2** the slope is **2(2) = 4**, as shown in Figure 9.7:

Or when **x = 5** the slope is **2(5) = 10**, and so on.

Note: sometimes f′(x) is also used for "the derivative of":

$f'(x) = 2x$

"The derivative of **f(x)** equals **2x**".

Example 9.5

Find the fourth derivative of the function $f(x) = y = 3x^4 + 2x^3 - 4x^2 + x + 5$

Solution

$f(x) = y = 3x^4 + 2x^3 - 4x^2 + x + 5$, then

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(3x^4 + 2x^3 - 4x^2 + x + 5) = 12x^3 + 6x^2 - 8x + 1$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}(4x^3) = 36x^2 + 12x - 8$$

$$f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}(36x^2 + 12x - 8) = 72x + 12$$

$$f^{(4)} = \frac{d^4y}{dx^4} = \frac{d}{dx}(72x + 12) = 72$$

What is $\frac{d}{dx} x^3$?

We know $\mathbf{f(x) = x^3}$, and can calculate $\mathbf{f(x+\Delta x)}$:

Start with: $\mathbf{f(x + \Delta x) = (x + \Delta x)^3}$

Expand $(x + \Delta x)^3$: $\mathbf{f(x+\Delta x) = x^3 + 3x^2 \Delta x + 3x (\Delta x)^2 + (\Delta x)^3}$

The slope formula: $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in **f(x+Δx)** and **f(x)**: $\frac{x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

Simplify (x^3 and $-x^3$ cancel): $\frac{3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$

Simplify further (divide through by Δx): $= 3x^2 + 3x\,\Delta x + (\Delta x)^2$

And then **as Δx tends towards 0** we get: $\frac{d}{dx} x^3 = 3x^2$

Differentiation is actually written as a **limit**:

$$f'(x) = \lim_{\Delta x 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

"The derivative of **f** equals **the limit as Δx goes to zero** of f(x+Δx) - f(x) over Δx"

Or sometimes the derivative is written like this.

$$\frac{d}{dx} = \lim_{dx \to 0} \frac{f(x + dx) - f(x)}{dx}$$

The process of finding a derivative is called "differentiation".

Below is the table containing basic rules which can be used to differentiate more complicated functions without using the differentiation from the first principles.

Name of rule	**f(x)**	**f′(x)**
Differentiating a constant C	C	0
Constant times a function	C[u(x)]	C[(u′(x)]
Differentiating x^n	x^n	nx^{n-1}
Sum or difference rule	u(x)±v(x)	u′(x)±v′(x)
Quotient rule	$\frac{u(x)}{v(x)}, v(x \neq 0)$	$\frac{u(x)}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$

Example 9.6

Let $f(x) = \frac{x^3 - 1}{x}$. Find $f'(3)$ and $f''(-4)$.

Solution

$$f'(x) = \frac{d}{dx}\left(\frac{x^3 - 1}{x}\right) = \frac{(x^3-1)'(x) - (x^3 - 1)(x)'}{x^2} = \frac{3x^2(x) - (x^3 - 1)(1)}{x^2} = \frac{3x^3 - x^3 + 1}{x^2} =$$

$$\frac{2x^3 + 1}{x^2} = 2x + \frac{1}{x^2}$$

$$f'(3) = 2(3) + \frac{1}{(3)^2} = 6 + \frac{1}{9} = \frac{54 + 1}{9} = \frac{55}{9}$$

$$f''(x) = \frac{d}{dx}\left(2x + \frac{1}{x^2}\right) = 2 + \frac{1'(x^2) - 1(x2)'}{4} = 2 + \frac{0 - 2x}{x^4} = 2 - \frac{2}{x^3}$$

$$f''(-4) = 2 - \frac{2}{(-4)^3} = 2 - \frac{2}{-64} = 2 + \frac{2}{64} = \frac{128 + 2}{64} = \frac{130}{64} = \frac{65}{32}$$

Example 9.7

Find f′(x) for f(x) equal to:

a) $x^3 + 2x^2 - 3x + 5$

b) $7x - \frac{4}{x} + \frac{3}{x^3}$

c) $3\sqrt{x} + \frac{2}{x}$

d) $x^2 - \frac{4}{\sqrt{x}}$

Solution

a) $f(x) = x^3 + 2x^2 - 3x + 5$

$f'(x) = 3x^2 + 2(2x) - 3(1) + 0 = 3x^2 + 4x - 3$

b) $f(x) = 7x - \frac{4}{x} + \frac{3}{x^3} = 7x - 4x^{-1} + 3x^{-3}$

$f'(x) = 7(1) - 4(-1x^{-2}) + 3(-3x^{-4}) = 7 + 4x^{-2} - 9x^{-4} = 7 + \frac{4}{x^2} - \frac{9}{x^4}$

c) $f(x) = 3\sqrt{x} + \frac{2}{x} = 3x^{1/2} + 2x^{-1}$

$f'(x) = 3\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 2(-1x^{-2}) = \frac{3}{2}x^{-\frac{1}{2}} - 2x^{-2} = \frac{3}{2\sqrt{x}} - \frac{2}{x^2}$

d) $f(x) = x^2 - \frac{4}{\sqrt{x}} = x^2 - 4(x^{-\frac{1}{2}})$

$f'(x) = 2x - 4\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) = 2x + 2x^{-\frac{3}{2}} = 2x + \frac{2}{\sqrt{x^3}} = 2x + \frac{2}{x\sqrt{x}}$

Example 9.8

Find the slope function of $f(x) = x^2 - \frac{4}{x}$ and hence find the slope of the tangent to the function at the point where $x = 2$.

Solution

$f(x) = x^2 - \frac{4}{x} = x^2 - 4x^{-1}$

$f'(x) = 2x - 4(-1x^{-2}) = 2x + 4x^{-2} = 2x + \frac{4}{x^2}$

Now $f'(2) = 2(2) + \frac{4}{(2)^2} = 4 + \frac{4}{4} = 4 + 1 = 5$

So, the tangent has slope of 5.

Example 9.9

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

Solution

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$, .

$\frac{dy}{dx}$ is the derivative or slope function of $y = 3x^2 - 4$ from which the slope at any point can be found. It is the instantaneous rate of change in y as x changes.

Application activity 9.4

1. Find the derivative function, $f'(x)$ given that $f(x)$ is given by:

   a) $2x^2 - 9$
   b) $x^2 + 3x - 2$
   c) $x^3 + 3x^2 + 4x - 1$
   d) $5x^4 - 6x^2$
   e) $\frac{3x-6}{x}$
   f) $\frac{2x-3}{x^2}$
   g) $\frac{x^3+5}{x}$
   h) $\frac{x^3+x-3}{x}$
   i) $\frac{1}{\sqrt{x}}$
   j) $(2x-1)^2$
   k) $(x+2)^3$

2. For each of the following functions, find $\frac{dy}{dx}$:

   a) $y = x^3 + 3x^3 - 1$
   b) $y = 5x^2$
   c) $y = \frac{1}{5x^2}$
   d) $y = 100\,x$
   e) $y = 15\,(x+1)$
   f) $y = 4x^3$

3. Differentiate each of the following with respect to x:

   a) $x + 2$
   b) $x\sqrt{x}$
   c) $(5-x)^2$
   d) $\frac{6x^3 - 9x^4}{3x}$
   e) $4x - \frac{1}{4x}$
   f) $y = x(x+1)(2x-5)$

4. Find the slope of the tangent to each of the following functions at an indicated value of x:

   a) $y = x^2$ at $x = 1$
   b) $y = \frac{8}{x^2}$ at $x = 9$
   c) $y = 2x^2 - 3x + 7$ at $x = -1$
   d) $y = \frac{2x^2-5}{x}$ at $x = 2$
   e) $y = \frac{x^2-4}{x^2}$ at $x = 4$
   f) $y = \frac{x^3-4x-8}{x^2}$ at $x = -1$

5. Find the slope function of f(x) where f(x) is given by:

a) $2x - \sqrt{x}$ b) $\sqrt[3]{x}$

c) $-\frac{2}{\sqrt{x}}$ d) $4\sqrt{x} + x$

e) $\frac{4}{\sqrt{x}} - 5$ f) $3x^2 - x\sqrt{x}$

g) $\frac{5}{x^2\sqrt{x}}$ h) $2x - \frac{3}{x\sqrt{x}}$

6. a) If $y = 3x - \frac{2}{x}$, find $\frac{dy}{dx}$ and interpret its meaning.

b) The position of a car moving along a straight road is given by $S = t^2 + 6t$ metres where t is the time in seconds. Find $\frac{dS}{dt}$ and interpret its meaning.

c) The cost of producing and selling x pens each week is given by $C = 989 + 4x + \frac{1}{400}x^2$ FRW. Find $\frac{dC}{dx}$ and interpret its meaning.

The chain rule

Composite functions are functions like $(3x^2 + 4x)^5$, $\sqrt{2x - 1}$ or $\frac{1}{x^2 + 2x + 3}$. These functions are made up of two simpler functions.

(a) $y = (3x^2 + 4x)^5$ is $y = u^5$ where $u = 3x^2 + 4x$

(b) $y = \sqrt{2x - 1}$ is $y = \sqrt{u}$ where $u = 2x - 1$

(c) $y = \frac{1}{x^2 + 2x + 3}$ is $y = \frac{1}{u}$ where $u = x^2 + 2x + 3$

Notice that in the example $(3x^2 + 4x)^5$, if $f(x) = x^5$ and $g(x) = 3x^2 + 4x$

then $f(g(x)) = f(3x^2 + 4x) = (3x^2 + 4x)^5$

All of these functions can be made up in this way where we compose a function of a function. Thus, these functions are called **composite functions**.

Consider the function $y = (2x + 1)^3$ which is really $y = u^3$ where $u = 2x + 1$.

o We see that $\frac{dy}{du} = 3u^2 = 3(2x + 1)^2$ and $\frac{du}{dx} = 2$

o When we expand, we obtain

$$y = (2x + 1)^3 = (2x)^3 + 3(2x)^2 1 + 3(2x)1^2 + 1^3 = 8x^3 + 12x^2 + 6x + 1$$

$$\frac{dy}{du} = 24x^2 + 24x + 6 = 6(4x^2 + 4x + 1) = 3(2x + 1)^2 \text{ x } 2 = \frac{dy}{du}\,\frac{du}{dx}$$

From the above example we derive the formula of the chain rule:

If $y = f(u)$ where $u = u(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 9.10

For each of the following, find $\frac{dy}{dx}$:

a) $y = (3x^2 + 4x - 5)^3$

b) $y = \frac{4}{\sqrt{1-2x}}$

Solution

a) $y = (3x^2 + 4x - 5)^3$

Let $y = u^3$ where $u = 3x^2 + 4x - 5$. Thus, $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 6x + 4$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(6x + 4) = 3(3x^2 + 4x - 5)^2(6x + 4)$$

b) $y = \frac{4}{\sqrt{1-2x}}$

Let $y = \frac{4}{\sqrt{u}} = 4u^{-\frac{1}{2}}$ where $u = 1 - 2x$. Thus, $\frac{dy}{du} = -2u^{-\frac{3}{2}} = -\frac{2}{\sqrt{u^3}}$ and $\frac{du}{dx} = -2$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{dy}{dx} = \left(-\frac{2}{\sqrt{u^3}}\right)(-2) = \frac{4}{\sqrt{u^3}} = \frac{4}{\sqrt{(1-2x)^3}}$$

In general, If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

Application activity 9.3

1. Find the derivative function for each of the following:

a) $y = (4x - 5)^2$

b) $y = \frac{1}{5 - 2x}$

c) $y = \sqrt{3x - x^2}$

d) $y = (1 - 3x)^4$

e) $y = 6(5 - x)^3$

f) $y = \sqrt[3]{2x^3 - x^2}$

g) $y = \frac{6}{(5x - 4)^2}$

h) $y = \frac{4}{3x - x^2}$

i) $y = 2(x^2 - \frac{2}{x})^3$

2. Find the slope of the tangent to each of the following at an indicated value of x:

(a) $y = \sqrt{1 - x^2}$ at $x = \frac{1}{2}$

b) $y = (3x + 2)^6$ at $x = -1$

c) $y = \frac{1}{(2x-1)^4}$ at $x = 1$

d) $y = 6x\sqrt[3]{1 - 2x}$ at $x = 0$

e) $y = \frac{4}{x + 2\sqrt{x}}$ at $x = 4$

f) $y = (x + \frac{1}{x})^3$ at $x = 1$

3. If $y = x^3$ then $x = y^{\frac{1}{3}}$, find

(a) $\frac{dy}{dx}$

(b) $\frac{dx}{dy}$ and hence show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

The product rule

If u(x) and v(x) are two functions of x and y = uv then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} \text{ or } y' = u'(x)(v(x) + u(x)v'(x)$$

Example 9.11

Using product rule, find $\frac{dy}{dx}$ for each of the following:

a) $y = \sqrt{x+1}(2x-1)^2$

b) $x^2(x^2 - 2x)^4$

Solution

a) $y = \sqrt{x+1}(2x-1)^2$. Let $u = \sqrt{x+1}$ and $v = (2x-1)^2$

$u' = \frac{4}{2\sqrt{x+1}}$ and $v' = 2(2x-1)2 = 4(2x-1)$. Then

$\frac{dy}{dx} = u'v + uv' = \frac{1}{2\sqrt{x+1}}(2x-1)^2 + \sqrt{x+1} \times 4(2x-1) = \frac{(2x-1)^2}{2\sqrt{x+1}} +$

$4\sqrt{x+1}(2x-1)$

b) $y = x^2(x^2 - 2x)^4$. Let $u = x^2$ and $v = (x^2 - 2x)^4$

$u' = 2x$ and $v' = 4(x^2 - 2x)^3(2x - 2)$. Then

$\frac{dy}{dx} = u'v + uv' = 2x(x^2 - 2x)^4 + x^2 \times 4(x^2 - 2x)^3(2x - 2)$

$= 2x(x^2 - 2x)^4 + 4x^2(x^2 - 2x)^3(2x - 2)$

The quotient rule

If $Q(X) = \frac{u(x)}{v(x)}$ then $Q'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$

Or if $y = \frac{u}{v}$ where u and v are functions of x then $\frac{dy}{dx} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

Example 9.12

Use the quotient rule to find $\frac{dy}{dx}$, for each of the following functions:

a) $y = \frac{2x+1}{x^2-3}$ b) $y = \frac{\sqrt{x}}{(1-2x)^2}$

Solution

a) $y = \frac{2x+1}{x^2-3}$. Let $u = 2x + 1$ and $v = x^2 - 3$

$u' = 2$ and $v' = 2x$

Thus, $\frac{dy}{dx} = \frac{uv - uv}{v^2} = \frac{2(x^2-3)-(2x+1)2x}{(x^2-3)^2} = \frac{2x^2 - 6 - 4x^2 - 2x}{(x^2-3)^2} = \frac{-2x^2-2x-6}{(x^2-3)^2}$

b) $y = \frac{\sqrt{x}}{(1-2x)^2}$. Let $u\sqrt{x} = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$u' = \frac{1}{2}x^{\frac{1}{2}}$ and $v' = 2(1-2x)^1(-2) = -4(1-2x)$. Then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\frac{1}{2}x^{\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4} = \frac{(1-2x)\left[\frac{1-2x}{2\sqrt{x}} + 4x\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^4} =$$

$$\frac{1-2x+8x}{2\sqrt{x}(1-2x)^3} = \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

Application activity 9.6

1. Using the product rule find $\frac{dy}{dx}$, for each of the following:

a) $y = x^2(2x-1)$ b) $y = x^2(x-3)^2$

c) $y = x^2\sqrt{3-x}$ d) $y = \sqrt{x}(x-3)$

e) $y = x^3\sqrt{x-1}$ f) $y = \sqrt{x}(x-x^2)^3$

2. Find the slope of the tangent to the following at the indicated value of x:

a) $y = x^4(1-2x)^2$ at $x = -1$ b) $y = (x+2)^2(x-3)$ at $x = 1$

c) $y = x\sqrt{1-2x}$ at $x = -4$ d) $y = x$ at $x^3\sqrt{5-x^2} = 1$

3. If $y = \sqrt{x}\,(3 - x)^2$ show that $\frac{dy}{dx} = \frac{(3 - x)(3 - 5x)}{2\sqrt{x}}$. Find the x-coordinates of all points $y = \sqrt{x}\,(3 - x)^2$ where the tangent is horizontal.

4. Use the quotient rule to find $\frac{dy}{dx}$ for each of the following functions:

a) $y = \frac{x}{x + 2}$ b) $y = \frac{x - 4}{x + 5}$ c) $y = \frac{x}{x^2 - 3}$

e) $y = \frac{x^2 - 3}{3x - x^2}$ f) $y = \frac{x}{\sqrt{1 - 3x}}$ g) $\frac{x}{\sqrt{x^2 + 2}}$

5. For each of the following; find the slope of the tangent to the indicated value of x:

a) $y = \frac{x}{1 - 2x}$ at $x = 1$ b) $y = \frac{x^3}{x^2 + 1}$ at $x = -1$

c) $y = \frac{\sqrt{x}}{2x + 1}$ at $x = 4$ d) $y = \frac{x^2}{\sqrt{x^2 + 5}}$ at $x = -2$

9.3 Applications of differentiation

Geometric interpretation of derivatives

If the function $y = f(x)$ is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope function; it is the rate of change of y with respect to x. Since $f''(x) = \frac{d^2y}{dx^2}$ is the derivative of the slope function, it is the rate of change of slope and is related to a concept called convexity (bending) of a curve.

If $x = t$ is time and if $y = s(t)$ is displacement function of moving object, then $s'(t) = \frac{ds}{dt}$ is the velocity function. The derivative of velocity i.e. the second derivative of the displacement function is $s''(t)$ or $\frac{d^2s}{dt^2}$; it is the rate of change of the velocity function, which is, the acceleration function.

Equations of tangent and normal to a curve

Tangent line to a curve of function

Consider a curve $y = f(x)$. If P is the point with x-coordinate a, then the slope of the tangent at this point is $f'(a)$. The equation of the tangent is by equating slopes and is

$\frac{y - f(a)}{x - a} = f'(a)$ or

$y - f(a) = f'(a)(x - a)$

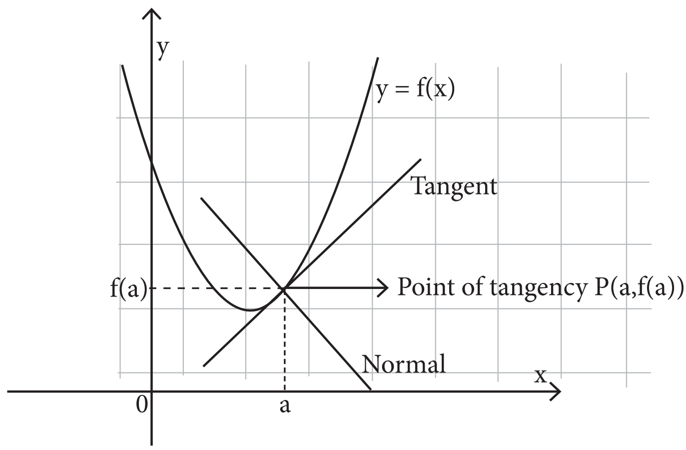


Fig. 9.8

Normal line to a curve of function

A normal to a curve is a line which is perpendicular to the tangent at the point of contact. Therefore, if the slope of the tangent at $x = a$ is $f'(a)$, then the slope of a normal at $x = a$ is $-\frac{1}{f'(a)}$. This comes from the fact that the product of gradients of two perpendicular lines is -1.

Note: If a tangent touches $y = f(x)$ at (a,b) then it has equation

$\frac{y-b}{x-a} = f'(a)$ or $y - b = f'(a)(x - a)$.

Vertical and horizontal lines have equations of the form $x = k$ and $y = c$ respectively, where c and k are constants.

Example 9.13

Find the equation of the tangent to $f(x) = x^2 + 2$ at the point where $x = 1$.

Solution

$f(x) = x^2 + 2$ and $f(1) = 1 + 2 = 3$, the point of contact is $(1, 3)$.

$f'(x) = 2x$ and $f'(1) = 2(1) = 2$

The tangent has equation $\frac{y-3)}{x-1} = 2$ $\quad y - 3 = 2(x - 1)$ $\quad y - 3 = 2x - 2$

$y = 2x + 1$.

Example 9.14

Find the equation of the normal to $y = \frac{8}{\sqrt{x}}$ at the point where $x = 4$.

Solution

When $x = 4$, $y = \frac{8}{\sqrt{4}} = \frac{8}{2} = 4$, so the point of contact is $(4, 4)$.

Then as $y = 8x^{-\frac{1}{2}}$ $\frac{dy}{dx} = -4x^{-\frac{3}{2}}$ and when $x = 4$, $\frac{dy}{dx} = -4\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}$, . The normal at (4, 4) has slope 2. So, the equation of the normal is

$\frac{y-4}{x-4} = 2$ $y - 4 = 2(x - 4)$ $y - 4 = 2x - 8y = 2x - 4$.

Example 9.15

Find the equation of the normal to $y = \frac{8}{\sqrt{x}}$ at the point where $x = 4$.

Solution

When $x = 4$, $y = \frac{8}{\sqrt{4}} = \frac{8}{2} = 4$, so the point of contact is (4, 4).

Find the equation of any horizontal tangent to $y = x^3 - 12x + 2$.

Solution

Let $f(x) = x^3 - 12x + 2$ then $f'(x) = 3x^2 - 12$

Horizontal tangents have gradient 0. So $3x^2 - 12 = 0$ $3(x^2 - 4) = 0$

$3(x + 2)(x - 2) = 0$. $x = -2$ or 2. Then

$f(2) = 8 - 24 + 2 = -14$ and $f(-2) = -8 + 24 + 2 = 18$

So points of contact are (2, –14) and (–2, 18).

Equations of tangents are $y = -14$ and $y = 18$.

Application activity 9.7

1. In each of the following questions, find, at the given point, the equation of the tangent and the equation of the normal.
   a) $y = x^2 - 4$ at $x = 1$ b) $y = x^2 + 4x - 2$ at $x = 0$
   c) $y = \frac{1}{x}$ at $x = -1$ d) $y = x^2 + 5$ at $x = 0$
   e) $y = x^2 - 5x + 7$ at $x = 2$ f) $y = (x - 2)(x^2 - 1)$ at $x = -2$
2. Find the equation of the tangent to each of the following functions at an indicated point:
   a) $y = x - 2x^2 + 3$ at $x = 2$ b) $y = \sqrt{x} + 1$ at $x = 4$
   c) $y = x^3 - 5x$ at $x = 1$ d) $y = \frac{4}{\sqrt{x}}$ at (1, 4)
3. Find the equation of the normal to each of the following functions at an indicated point:
   a) $y = x^2$ at the point (3, 9) b) $y = x^3 - 5x + 2$ at $x = -2$
   c) $y = \frac{5}{\sqrt{x}} - \sqrt{x}$ at the point (1, 4) d) $y = 8\sqrt{x} - \frac{1}{x^2}$ at $x = 1$

4. a) For the curve of the function $y = 2\sqrt{x} + \frac{1}{\sqrt{x}}$, find all points of contact of horizontal tangents.

   b) For the curve of the function $y = 2x^3 + 3x^2 - 12x + 1$, find the equations of all possible horizontal tangents to the curve.

   c) If the tangent to $y = 2x^3 + mx^2 - 3$ at the point where $x = 2$ has slope 4, find the value of m

   d) Find the equation of the tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at (1, 2).

5. Find the equation of the tangent to the following functions at an indicated point:

   a) $y = \sqrt{2x+1}$ at $x = 4$ b) $y = \frac{1}{2-x}$ at $x = -1$

   c) $f(x) = \frac{1}{1-3x}$ at $(-1, \frac{1}{4})$ d) $f(x) = \frac{x^2}{1-x}$ at $(2, -4)$

6. Find the equation of the normal to the following function at an indicated point:

   a) $y = \frac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$ b) $y = \frac{1}{\sqrt{3-2x}}$, at $x = -3$

   c) $f(x) = \sqrt{x}\,(1-x)^2$ at $x = 4$ d) $f(x) = \frac{x^2-1}{2x+3}$ at $x = -1$

7. Given the function $y = m\sqrt{1-nx}$ where m and n are constants, and has tangent with equation $3x + y = 5$ at the point where $x = -1$. Find m and n.

Mean value theorems

If f is continuous over a closed interval [a,b] and differentiable on the open interval (a,b), and if f(a) = f(b), then, there is at least one number c in]a,b[such that $f'(c)=0$.

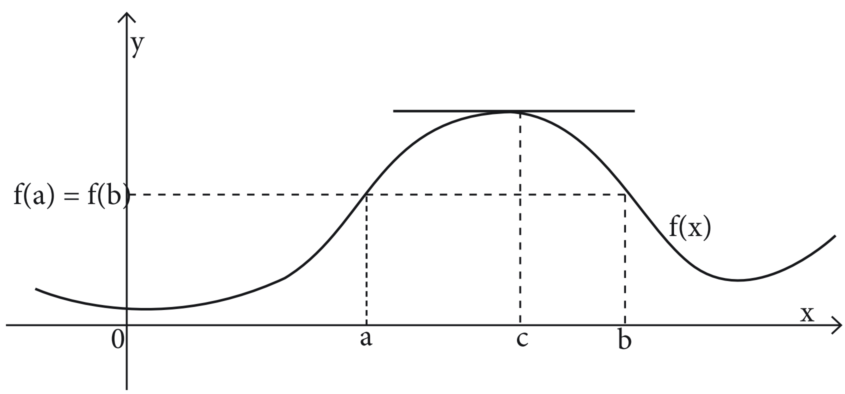


Fig. 9.9

Example 9.16

Using Rolle's theorem, find the point on the curve of $y = x^2$, $x \in [-2,2]$, where the tangent is parallel to the x-axis.

Solution

Here $f(x) = x^2$

(i) Since f(x) is a polynomial, therefore it is a continuous function for every value x and hence in particular it is continuous in the closed interval [–2, 2].

(ii) And, f(x) being a polynomial is derivable for each value of x and in particular in the open interval (–2, 2).

$f'(x) = 2x$ which exists in (–2, 2)

(iii) Also, $f(-2) = (-2)^2 = 4$ and $f(2) = (2)^2 = 4$

$f(-2) = f(2)$

Thus f(x) satisfies all the three conditions of Rolle's theorem and therefore there must exist at least one real value c in (-2, 2)

$f'(c) = 0 \quad 2c = 0 \quad c = 0 \in (-2, 2)$

Now equation of the curve is $y = x^2$. Then $x = 0$,

$y = 0$ at (0, 0) the tangent to the curve $y = x^2$ is $T \equiv y - 0 = 0(x - 0)$ $T \equiv y = 0$ which is the x-axis itself.

Activity 9.3

1) Verify Rolle's theorem for the function $f(x) = x(x - 3)^2$, $0 \le x \le 3$

2) Verify Rolle's theorem for the function $f(x) = 2(x + 1)(x - 2)$ defined in the interval [–1, 2].

Lagrange's mean value theorem

If f(x) is continuous on the closed interval [a,b] and differentiable on the open interval]a,b[, then there exist at least one number c in]a,b[such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

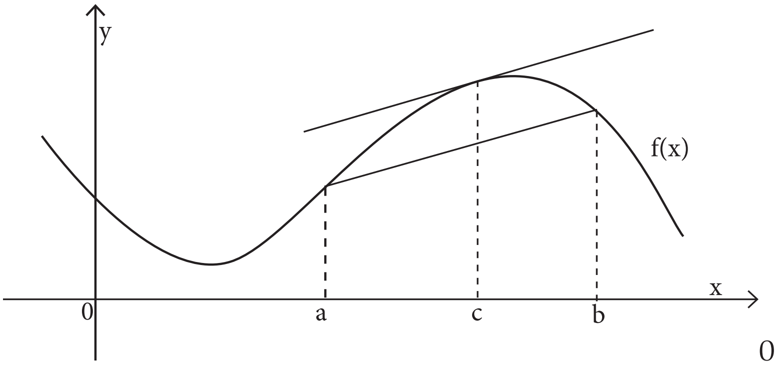


Fig. 9.10

Example 9.17

Verify Lagrange's Mean Value theorem of the function (x) = $x^3 - 2x^2 - x + 3$ defined in the interval [0, 1].

Solution

$f(x) = x^3 - 2x^2 - x + 3$

f(x) being a polynomial function is continuous and derivable for all values of x.

f(x) is continuous on [0, 1] and derivable on (0, 1).

Thus, f(x) satisfies both the conditions of Mean Value Theorem.

There must be at least one number c such that $0 < c < 1$

$$\frac{f(1) - f(0)}{1 - 0} = f'(c)$$

Now $f(1) = (1)^3 - 2(1)^2 - 1 + 3 = 1$

$f(0) = (0)^3 - 2(0)^2 - 0 + 3 = 3$

$f'(x) = 3x^2 - 4x - 1$

$f'(c) = 3(c)^2 - 4c - 1$

$\frac{1 - 3}{1} = 3c^2 - 4c - 1 \qquad 3c^2 - 4c - 1 = -2 \qquad 3c^2 - 4c + 1 = 0$

$(3c - 1)(c - 1) = 0$

$c = \frac{1}{3}$ or $c = 1$

But $c = 1 \notin (0,1)$ whereas $c = \frac{1}{3} \in (0,1)$

Hence Mean Value Theorem is verified.

L'Hôpital theorem

This is a rule for evaluating indeterminate forms. One of the forms of the rule is the following:

Theorem: Suppose that $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, if the limit on the right-hand side exists.

Example 9.18

Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$

Solution

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{0}{0} \text{ (indeterminate form)}$$

The result also holds if $f(x) \to \infty$ and $g(x) \to \infty$ and as $x \to a$.
Moreover, the results apply if '$x \to a$' is replaced by '$x \to +\infty$' or '$x \to \infty-$'.

Application activity 9.8

Use l'Hoppital's rule (where applicable) to find the following limits:

1. $\lim_{x \to -\infty} \frac{4x+3}{3x^2+5}$
2. $\lim_{x \to 3} \frac{9-x^2}{x^3-3x^2+2x-6}$
3. $\lim_{x \to -1} \frac{x^3-2x-1}{3x^2-2x-5}$
4. $\lim_{x \to \infty} \frac{x^3-7x^4}{3x^4-5x^3}$
5. $\lim_{x \to 2} \frac{4-x^2}{x^2-3x+2}$

Increasing and decreasing functions

A real function f is **increasing** in or on an interval I if $f(x_1) \le f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is **strictly increasing** if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
A real function f is **decreasing** in or on an interval I if $f(x_1) \ge f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is **strictly decreasing** if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

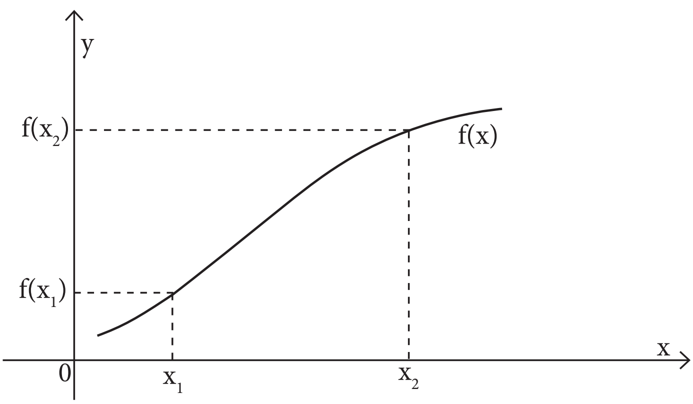


Fig. 9.11: Increasing function

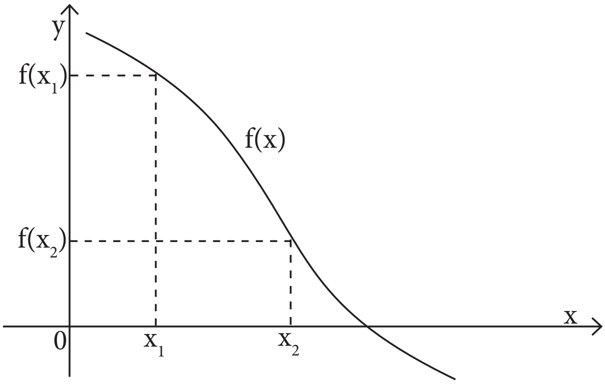


Fig. 9.12: Decreasing function

Meaning of the sign of the derivative

If we recall that the derivative of a function yields the slope of the tangent to the curve of the function. It appears that a function is increasing at a point where the derivative is positive and decreasing where the derivative is negative.

Example 9.19

Given the function $f(x) = 2x^3 + 9x^2 + 12x + 20$. Determine the interval where the graph of the function is increasing and where it is decreasing.

Solution

$f(x) = 2x^3 + 9x^2 + 12x + 20$

$f'(x) = 6x^2 + 18x + 12$

$f'(x) = 0 \Rightarrow 6x^2 + 18x + 12 = 0$

$6(x^2 + 3x + 2) = 0$

$6(x + 2)(x + 1) = 0$

$x = -2$ 0r $x = -1$

We have the following summary table of sign of $f'(x)$

x / Function	$-\infty$		-2		-1		$+\infty$
$x + 2$		$-$	0	$+$		$+$	
$x + 1$		$-$		$-$	0	$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		↗		↘		↗	

Fig. 9.13

Thus, $f(x)$ is strictly increasing on $]-\infty, -2[\cup]-1, +\infty[$

$f(x)$ is strictly decreasing on $]-2, -1[$

Stationary point

This is a point on the graph $y = f(x)$ at which f is differentiable and $f'(x) = 0$. The term is also used for the number c such that $f'(c)$. The corresponding value f(c) is a stationary value. A stationary point c can be classified as one of the following, depending on the behaviour of f in the neighbourhood of c:

(i) A local maximum, if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c,

(ii) A local minimum, if $f'(x) < 0$ to the left of c and $f'(c) > 0$to the right of c,

(iii) Neither local maximum nor minimum, if (i) and (ii) are not satisfied.

Note: Maximum and minimum values are termed as extreme values.

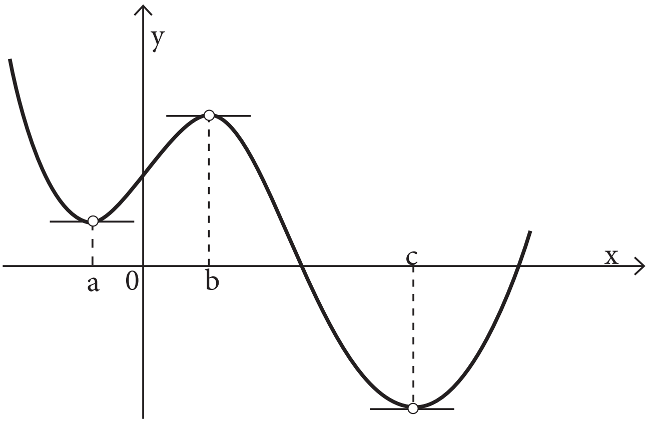


Fig. 9.14

The points a, b and c are stationary points.

Example 9.20

Find the stationary point of the function defined by $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$

Solution

$f'(x) = x^2 + x - 2$

$f'(x) = 0 \Rightarrow x^2 + x - 2 = 0$

$(x - 1)(x + 2) = 0$

$x = 1$ or $x = -2$

The stationary points of the function $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$ are found at $x = 1$ and $x = -2$ and are $(1, -1.17)$ and $(-2, 3.33)$.

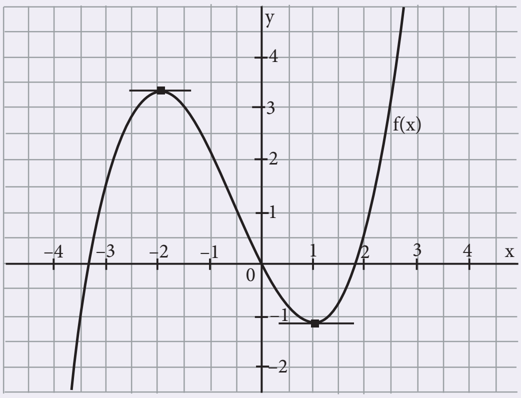


Fig. 9.15

Point of inflection

A point of inflection is a point on a graph $y = f(x)$ at which the concavity changes.

If f' is continuous at a , then for $y = f(x)$ to have a point of inflection at a it is necessary that $f''(a) = 0$, and so this is the usual method of finding possible points of inflection.

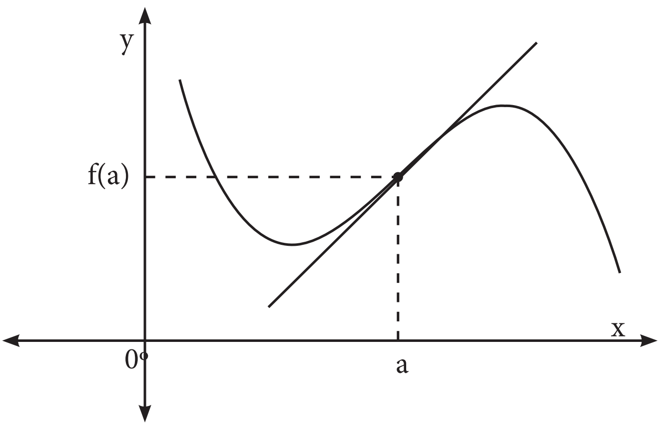


Fig. 9.16

However, the condition $f''(a) = 0$ is not sufficient to ensure that there is a point of inflection at a; it must be shown that $f''(x)$ is positive to one side of a and negative to other side.

Thus, if $f(x) = x^4$ then $f''(0) = 0$; but $y = x^4$ does not have a point of inflection at 0 since $f''(x)$ is positive to both sides of 0.

Example 9.21

Find the inflection point of the function defined by $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$

Solution

$f'(x) = x^2 + x - 2$

$f''(x) = 2x + 1$

$f''(x) = 0 \Leftrightarrow 2x + 1 = 0$

$x = -\frac{1}{2}$

The inflection point to the curve is at $x = -\frac{1}{2}$ and is (–0.5, 1.08).
Below is the graph illustrating the inflection point:

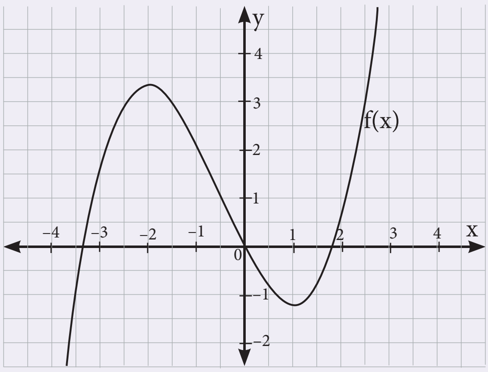


Fig. 9.17

Concavity

At a point of graph $y = f(x)$, it may be possible to specify the concavity by describing the curve as either concave up or concave down at that point, as follows:

Fig. 9.18

1. A curve is said to be concave downwards (or concave) in an interval $]a, b[$ If $f''(x) < 0$ for all $x \in]a,b[$.
2. A curve is said to be concave upwards (or convex) in an interval $]a, b[$ if $f''(x) > 0$ for all $x \in]a,b[$.

Example 9.22

Given the function $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

(a) State the values of x for which f is increasing

(b) Find the x-coordinate of each extreme point of f.

(c) State the values of x for which the curve of f is concave upwards.

(d) Find the x-coordinate of each point of inflection.

(e) Sketch the general shape of the graph of f indicating the extreme points and points of inflection.

Solution

(a) $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

$f'(x) = x^2 + x = x(x + 1)$

Sign of f′(x)

x / Function	$-\infty$		-1		0		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		↗		↘		↗	

Fig. 9.19

Therefore f is increasing for $x < -1$ or $x > 0$

(b) The extreme points are at $x = -1$ and $x = 0$.

(c) $f'(x) = x^2 + x$

The sign of f′′(x)

$f''(x) = 2x + 1$

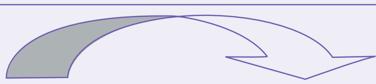
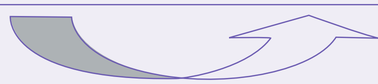
x / Function	$-\infty$		$-\frac{1}{2}$		$+\infty$
$f''(x)$		$-$	0	$+$	
$f(x)$					

Fig. 9.20

The curve is concave upwards for $x > -\frac{1}{2}$

(d) $x = -\frac{1}{2}$ is a point of inflection

(e) The graph of $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

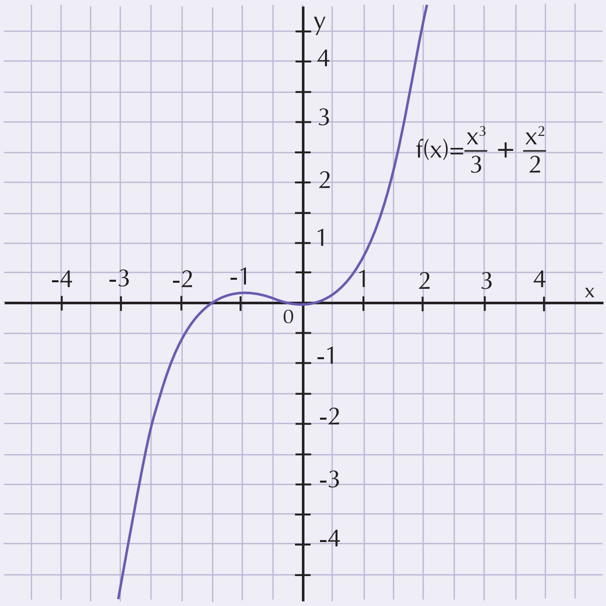


Fig. 9.21

Application activity 9.9

In groups of four, draw the following graphs. For each, where applicable:

i. state the values of x for which f is increasing

ii. find the x-coordinate of each extreme point of f.

iii. state the values of x for which the curve of f is concave upwards.

iv. find the x-coordinate of each point of inflection

v. indicate the extreme points and points of inflection.

(a) $f(x) = x^3 - 3x^2$.

(b) $f(x) = x^4 - 4x^3$

(c) $f(x) = x^2 (x - 2)^3$

Rate of change problems

Mental task

We can use differentiation to help us solve many rate of change problems. Can you mention any such problems?

Gradient as a measure of rate of change

If $y = f(x)$ is a differentiable function of x, then $\frac{dy}{dx}$ is the rate of change of y with respect to x.

Example 9.23

A particle moves along the curve $y = \frac{1}{2}x^2$. Find the points on the curve at which the y-coordinate is changing twice as fast the x-coordinate.

Solution

$y = \frac{1}{2}x^2 - 7$

$\frac{dy}{dx} = x$. and we are given $\frac{dy}{dx}$. Equating the two gives

$x = 2$

When $x = 2$ $\quad y = \frac{1}{2}(2)^2 - 7 = 2 - 7 = -5$

Therefore, the required point on the curve is: $(2, -5)$.

Application activity 9.10

1. The side of a square sheet of metal is increasing at 4 cm per second. At what rate is the area increasing when the side is 8 cm long?
2. The radius of a circle is increasing uniformly at the rate of 3 cm per second. At what rate is the area increasing when the radius is 10 cm?
3. An oil storage tank is built in the form of an inverted circular cone with height of 6 cm and base radius of 2 m. Oil is being pumped into the tank at the rate of 2 litres per minute = 0.002 m^3 per min. How fast is the level of oil rising when the tank is filled to a height of 3 m?
4. An inverted circular cone has a depth of 10 cm and base radius of 5 cm. Water is poured into it at the rate of 1.5 cm^3 per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm?
5. A company that manufactures dog food wishes to pack the food in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of 128π cm^3 and the minimum possible surface area

Kinematic meaning of derivatives

Motion of a body on a straight line

Consider a body moving along the x-axis such that its displacement, x metres on the right of the origin 0 after a time t seconds ($t \geq 0$), is given by $x = f(x)$.

The average velocity of the body in the time interval $[t, t+h]$ is given by

$$\vec{v} = \frac{\text{total displacement}}{\text{total time taken}} = \frac{f(t+h)-f(t)}{h} \quad (h \neq 0).$$

In order to find the instantaneous velocity of the body at time t seconds, we find the average velocity in the time interval [t, t+h] and let h take smaller and smaller values. In fact the instantaneous velocity of the body at time t seconds is defined to be

$$v(t) = \lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$$

This is the first derivative of the function $x = f(t)$ and so $v(t) = \frac{dx}{dt} = f'(t)$.

Then differentiating with respect to time the 'dot' notation is often used.

Thus, if $x = f(t)$, then $v(t) = \frac{dx}{dt} = \dot{x} = f'(x)$

We define the **velocity** to be the rate of change of displacement.

Acceleration is defined as the rate of change of velocity. Thus the acceleration of the body moving along a straight line with displacement x(t) at time t and is given by

$$a(t) = \frac{d}{dx}(v(t)) = v'(t) = \frac{d^2}{dt^2}(x(t)) = x''(t) = \ddot{x}(t).$$

Velocity is a vector quantity and so the direction is critical. If the body is moving towards the right (the positive direction of the x-axis), its velocity is positive and if it is moving towards the left, its velocity is negative. Therefore, the body changes motion when velocity changes sign. A sign diagram of the velocity provides a deal with information regarding the motion of the body.

Example 9.24

A body moves along the x-axis so that at time t seconds $x(t) = t^3 + 3t^2 - 9t$. Find:

(a) the position and velocity of the body at t = 0. 1, 2

(b) where and when the body comes to rest

(c) the maximum speed of the body in the first 1 second of motion

(d) the maximum velocity of the body in the first 1 second of motion

(e) the total distance travelled by the body in the first 2 seconds of motion.

Solution

(a) $x(t) = t^3 + 3t^2 - 9t$ $\qquad v(t) = 3t^2 + 6t - 9 = 3(t^2 + 2t - 3) = 3(t + 3)(t - 1)$

When t = 0, x = 0 and v = –9; when t = 1, x = –5 and v = 0; when t = 2, x = 2 and v = 5 .

At t = 0, the body is at the origin with velocity of $-9ms^{-1}$.

At t = 1, the body is 5m to the left of 0 with velocity of $0ms^{-1}$.

At t = 2, the body is 2m to the right of 0 with velocity of $15\ ms^{-1}$.

(b) The body is at rest when $v = 0$. This occurs when $t = 1$ $(t \geq 0)$.
At this time the body is 5m to the left of the origin.

(c) The velocity is increasing in the interval $[0, 1]$ since $v'(t) = 6t + 6 > 0$.
$v(0) = -9$ and $v(1) = 0$.
Therefore the maximum speed in the first 1 second is $9ms^{-1}$.

(d) From part (c), the maximum velocity is $0ms^{-1}$.

(e) The following diagram illustrates the position of the body from $t = 0$ to $t = 2$.

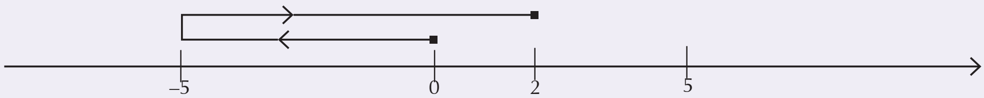


Fig. 9.22

From the diagram the total distance travelled is 12 m.

Application activity 9.11

1. Write down the first and the second derivatives of each of the following:

   (a) $5x - 4$ (b) $3x^2 - 6x - 5$

   (c) $2x^3 - 5x^2 + 4x + 2$ (d) $x^3 - \frac{2}{x}$

   (e) $\frac{3x + 2}{\sqrt{x}}$ (f) $\frac{6}{x} - \frac{3}{x^2} + \frac{4}{x^3}$

2. A body moves along the x-axis so that its position is $x(t)$ metres to the right of the origin at time t seconds.

   (a) If $x(t) = t^3 - 3t^2$ explain why the total distance travelled in the first three seconds of motion is not equal to the displacement in that time.

   (b) If $x(t) = t^3 - 3t^2 + 3t$ explain why the distance travelled in that first three seconds of motion is now equal to the displacement in that time.

3. A particle is moving along the x-axis such that its position, $x(t)$ metres to the right of the origin at time t seconds, is given by $x(t) = t^3 - 9t^2 + 24t - 18$. Describe the particle motion during the first five seconds and calculate the distance travelled in that time.

4. A ball is thrown vertically into the air so that it reaches a height of $y = 19.6t - 4.9t^2$ metres in t seconds.

   (a) Find the velocity and acceleration of the ball at time t seconds.

   (b) Find the time taken for the ball to reach its highest point.

   (c) How high did the ball rise?

   (d) At what time(s) would the ball be at half its maximum height?

Optimization problems

Activity 9.4

Carry out research on the meaning of optimization. Where is optimization applied?

Mathematical optimization is the selection of a best element (with regard to some criteria) from some set of available alternatives.

In the simplest case, an optimization problem consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set and computing the value of the function.

Example 9.25

We are enclosing a rectangular field with 500 m of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.

Solution

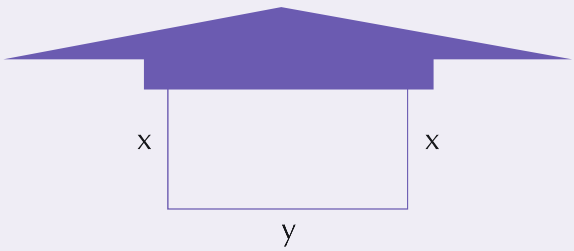


Fig. 9.23

Maximize the area $A = xy$ subject to constraint of $y + 2x = 500$.

Solve constraint for y and substitute into area.

$y = 500 - 2x$

$A = xy = x(500 - 2x) = 500x - 2x^2$

Differentiate and find critical point(s).

$A = 500 - 4x \Rightarrow 500 - 4x = 0 \Rightarrow x = 125$

By second derivative test this is a relative maximum and so is the answer we are after. Finally, find y.

$y = 500 - 2x = 500 - 2(125) = 250$

The dimensions are 125 m x 250 m

An optimization problem can be represented in the following way:

Given: a function $f: A \to R$ from some set A to the real numbers

Sought: an element x_0 in A such that $f(x_0) \leq f(x)$ for all x in A ("minimization") or such that $f(x_0) \geq f(x)$ for all x in A ("maximization").

The theory of maxima and minima can be applied to a great variety of problems.

Example 9.26

1. Determine two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution

1. Let one number be x , then the other is (15 – x)

   Since the numbers are positive, $0 < x < 15$

   Let S be the sum of the squares of the numbers,

   $S = x^2 + (15 - x)^2 = x^2 + 225 - 30x + x^2 = 2x^2 - 30x + 225$

   $\frac{dS}{dx} = 4x - 30$

   For maximum or minimum, $\frac{dS}{dx} = 0$

   $4x - 30 = 0$

   $4x = 30; \ x = \frac{15}{2}$

   $\frac{d^2s}{dx^2} = 4 > 0,$

   Hence S is minimum, when $x = \frac{15}{2}$

   Other number $= 15 - \frac{15}{2} = \frac{30 - 15}{2} = \frac{15}{2}$

   Hence, the sum of squares is minimum when the numbers are $\frac{15}{2}$ and $\frac{15}{2}$.

Application activity 9.12

1. Find the maximum profit that a company can make, if the profit function is given by $P(x) = 4 + 24x - 18x^2$
2. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Summary

1. For a **non-linear function** with equation $y = f(x)$, slopes of tangents at various points continually change.
2. The **derivative** of a function also known as slope of a function, or derived function or simply the derivative is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

3. The **slope of the tangent** at the point $x = a$ is defined as the slope of the curve at the point where $x = a$, and is the instantaneous rate of change in y with respect to x at that point.
4. If f is a function which is differentiable on its domain, then f' is a function. If, in addition, f' is differentiable on its domain then the derivative of f' exists and is denoted by f''; it is the function given by $f''(x) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$ and is called the second derivative of f.
5. **Differentiation** is the process of finding the derivative function.
6. If $Q(x) = \frac{u(x)}{v(x)}$, then $Q'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$
7. If the function $y = f(x)$ is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope function; it is the **rate of change** of y with respect to x.
8. A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.
9. **Rolle's Theorem:** If f is continuous over a closed interval [a, b] and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then, there is at least one number c in]a, b[such that $f'(c) = 0$.
10. If f(x) is **continuous** on the closed interval [a, b] and **differentiable** on the open interval]a, b[, then there exist at least one number c in]a, b[such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
11. **L'Hфpital's theorem:** A rule for evaluating indeterminate forms. $\frac{0}{0}$ and $\frac{\infty}{\infty}$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

12. **Stationary point:** It is a point on the graph $y = f(x)$ at which f is differentiable and $f'(x) = 0$
13. A curve is said to be **concave downwards** (or concave) in an interval]a, b[If $f''(x) < 0$ for all $x \in]a, b[$.
14. A curve is said to be **concave upwards** (or convex) in an interval]a, b[if $f''(x) > 0$ for all $x \in]a, b[$.

Topic area: Linear algebra

Sub-topic area: Vectors

Unit 10

Vector space of real numbers

Key unit competence

Determine the magnitude and angle between two vectors and to be able to plot these vectors. Also, be able to point out the dot product of two vectors.

10.0 Introductory activity

Given the point A with coordinates $(1,2)$ and the point B with coordinates $(-3,1)$.

i) Represent the points A and B in the same plane XY

ii) Draw arrow from point A to point B

iiii) Find the components of the vector $\vec{AB}$

10.1 Vector spaces $\mathbb{R}^2$

Definitions and operations on vectors

In Senior 2, we were introduced to vector and scalar quantities.

Application activity 10.1

1. Define a vector.
2. Differentiate between a vector and a scalar quantity.
3. Use diagrams to illustrate equal vectors.

A quantity which has both **magnitude** and **direction** is called a **vector**. It is usually represented by a directed line segment. In our daily life, we deal with two mathematical quantities:

(a) one which has a defined magnitude but for which direction has no meaning (example: length of a piece of wire).

In physics, length, mass and speed have magnitude but no direction. Such quantities are defined as **scalars**.

(b) the other for which direction is of fundamental significance. Quantities such as force and wind velocity depend very much on the direction in which they act. These are **vector** quantities.

Addition of vectors

Vectors are added by the triangle law or end-on rule or parallelogram law of addition.

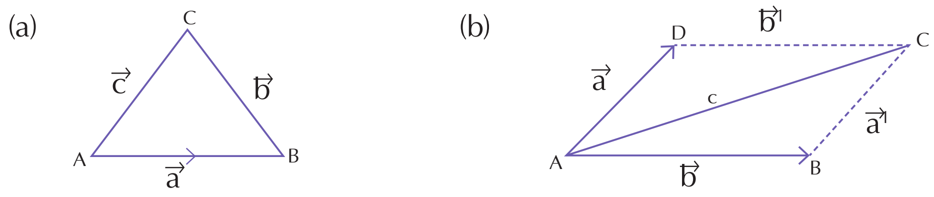


Fig. 10.1

In Figure 10.1(a) vector $\vec{a} = \overrightarrow{AB}$ followed by vector $\vec{b} = \overrightarrow{BC}$ giving a vector $\vec{c} = \overrightarrow{AC}$.

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e $\vec{c} = \vec{a} + \vec{b}$

In Figure 10.1(b) ABCD is a parallelogram and $\overrightarrow{BC} = \overrightarrow{AD}$.

$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AD} + \overrightarrow{AC}$ i.e $\vec{c} = \vec{a} + \vec{b}$

The parallelogram law

Also $\overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC} = \vec{a} + \vec{b}$. Hence, $\overrightarrow{AC} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$

Vector addition is commutative i.e vectors can be added in any order. The displacement $\overrightarrow{AC}$ is the same as the displacement $\overrightarrow{AB}$ followed by $\overrightarrow{BC}$.

Vector subtraction

The vector $\vec{a} - \vec{b}$ is equal to $\vec{a} + (-\vec{b})$ thus $\vec{a} - \vec{b}$ can be found by adding vector $-\vec{b}$ to vector $\vec{a}$.

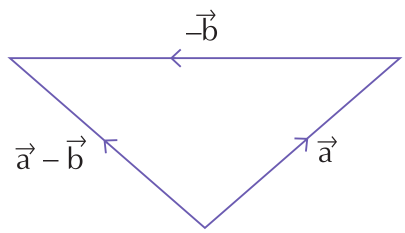


Fig. 10.2

Example 10.1

Use Figure 10.3 to simplify $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$.

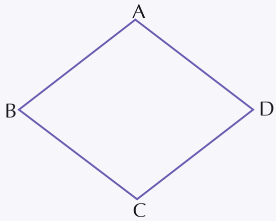


Fig. 10.3

Solution

$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

Addition and subtraction using column vector

The displacement from A to B followed by displacement B to C is equivalent to the displacement from A to C, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Example 10.2

If $\overrightarrow{AB} = (5, 4)$ and $\overrightarrow{BC} = (6, -2)$, find $\overrightarrow{AC}$.

Solution

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}$

Multiplication of a vector by a scalar

When a vector is multiplied by a scalar (a number) its magnitude changes. If it is multiplied by a positive number the direction remains the same.

However, if it is multiplied by a negative number, the direction of the vector reverses.

Vector $\vec{a}$ multiplied by 2 is the vector $2\vec{a}$. It is twice the magnitude (and positive) and in the same direction. Vector $\vec{b}$ multiplied by -1 is the vector $-\vec{b}$, same magnitude but opposite in the direction.

Example 10.3

Given that A(2, 1), B(4, 4) and C(6, 7), find $\overrightarrow{AC}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Solution

$\overrightarrow{AB} = (2, 3)$ and $\overrightarrow{BC} = (2, 3)$

$\overrightarrow{AC} = (4, 6) = 2(2, 3) = 2\overrightarrow{AB} = 2\overrightarrow{BC}$.

Note that multiplication of a vector by a scalar (i.e a number) is performed by multiplying each element of the vector. If a vector $\overrightarrow{AB} = (a, b)$ then $k\overrightarrow{AB} = k(a, b) = (ka, kb)$.

10.2 Vector spaces of plane vectors ($\mathbb{R}$, V, +)

The definition of vector space denoted by V needs the arbitrary fields F = $\mathbb{R}$ whose elements are called **scalars.**

Definitions and operations

A vector space is a set V whose elements are called vectors, together with a set of scalars $\mathbb{R}$ (but in general, this set of scalars can be any field F.) We have two operations: vector addition and scalar multiplication, which satisfy the following 10 axioms, for all $\vec{x}, \vec{y}, \vec{z} \in V$ and r, s, $\in \mathbb{R}$:

1. $\vec{x} + \vec{y} \in V$. (Closure under vector addition.)
2. $\vec{x} + \vec{y} = \vec{x} + \vec{y}$. (Commutativity)
3. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$. (Associativity of vector addition.)
4. There exists a $\vec{0} \in V$ such that $\vec{0} + \vec{x} = \vec{x} + \vec{0} = \vec{0}$. (Existence of additive identity.)
5. There exists $-\vec{x} \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$. (Existence of additive inverse.)
6. $r\vec{x} \in V$. (Closure under scalar multiplication.)
7. $r(\vec{x} + \vec{y}) = r\vec{x} + r\vec{y}$. (Distributivity of vector sums.)
8. $(r + s)\vec{x} = r\vec{x} + s\vec{x}$. (Distributivity of scalar sums)
9. $r(s\vec{x}) = (rs)\vec{x}$. (Associativity of scalar multiplication.)
10. $1\vec{x} = \vec{x}$. (Scalar multiplication identity.)

Properties of vectors

a) The sum of an infinite list of vectors $\vec{v}_1; \vec{v}_2 ..., \vec{v}_k$ can be computed in any order, and fully parenthesized in any way, and the sum will be the same.

b) $\vec{v} + \vec{w} = \vec{0}$ if and only if $w = -\vec{v}$.

c) The negation of $\vec{0}$ is $\vec{0}$: $-\vec{0} = \vec{0}$.

d) The negation of the negation of a vector is the vector itself: $-(-\vec{v}) = \vec{v}$.

e) If $\vec{v} + \vec{z} = \vec{v}$, then $\vec{z} = \vec{0}$. Thus, $\vec{0}$ is the only vector that acts like $\vec{0}$.

f) Zero times any vector is the zero vector: $0\vec{v} = \vec{0}$ for every vector $\vec{v}$.

g) Any scalar times the zero vector is the zero vector: $c\vec{0} = \vec{0}$ for every real number *c*.

h) The only ways that the product of a scalar and a vector can equal the zero vector are when either the scalar is 0 or the vector is $\vec{0}$. That is, if $c\vec{v} = \vec{0}$, then either $c = 0$ or $\vec{v} = \vec{0}$.

i) The scalar –1 times a vector is the negation of the vector: $(-1)\vec{v} = -\vec{v}$.

We define subtraction in terms of addition by defining $\vec{v} - \vec{w}$ as an abbreviation for $\vec{v} + (-\vec{w})$.

$\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$

All the usual properties of subtraction follow, such as

j) $\vec{u} + \vec{v} = \vec{w}$ if and only if $\vec{u} = \vec{w} - \vec{v}$.

k) $c(\vec{v} - \vec{w}) = c\vec{v} - c\vec{w}$.

l) $(c - d)\vec{v} = c\vec{v} - d\vec{v}$

Example 10.4

Show that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y): x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

The vector addition in $\mathbb{R}^2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$

$(x_1, y_1)\, x(x_2,y_2) \to (x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$

The scalar multiplication $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$

$(\alpha(x_1, y_1)) \to \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

Solution

Let $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$, $\vec{w} = (x_3, y_3) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$:

(i) We verify if $(\mathbb{R}^2,+)$ is an abelian group, that is:

$\vec{u} + \vec{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$ (closure)

1. $(\vec{u} + \vec{v}) + \vec{w} = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$
$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3) = (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$
$= \vec{u} + (\vec{v} + \vec{w})$ (Associativity for addition)

2. $\vec{u} + \vec{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = (x_2, y_2)$
$+ (x_1, y_1) = \vec{v} + \vec{u}$; (Commutativity for addition)

3. $\vec{u} + \vec{0} = (x_1, y_1) + (0, 0) = (x_1, y_1) = \vec{u}$ (Additive identity is (0, 0))

4. $\vec{u} + (-\vec{u}) = (x_1, y_1) + (-x_1, -y_1) = (x_1 - x_1, y_1 - y_1) = (0, 0) = \vec{0}$
$((x_1, y_1)$ (has an additive inverse $(-x_1, -y_1))$

(ii) We verify if the scalar multiplication satisfies the following:

1. $\alpha(\beta\vec{u}) = \alpha[\beta(x_1, y_1)] = \alpha(\beta x_1, \beta y_1) = (\alpha\beta x_1, \alpha\beta y_1) = \alpha\beta(x_1, y_1) = \alpha\beta(\vec{u})$

   (Associativity for multiplication)

2. $1\vec{u} = 1(x_1, y_1) = (1x_1, 1y_1) = (x_1, y_1) = \vec{u}$ (Multiplicative identity is 1)

3. $\alpha(\vec{u} + \vec{v}) = \alpha[(x_1, y_1) + (x_2, y_2)] = \alpha(x_1 + x_2, y_1 + y_2) = (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$

   $= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2) = \alpha(x_1, y_1) + \alpha(x_2, y_2) = \alpha\vec{u} + \alpha\vec{v}$

   (Right distributivity)

4. $(\alpha + \beta)\vec{u} = (\alpha + \beta)(x_1,y_1) = [(\alpha + \beta)x_1, (\alpha + \beta)y_1] = (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_1)$

   $= (\alpha x_1, \alpha y_1) + \beta x_1, \beta y_1) = \alpha(x_1, y_1) + \beta(x_1, y_1) = \alpha\vec{u} + \beta\vec{u}$

   (Left distributivity)

Hence, $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

Application activity 10.2

1. Let f($\mathbb{R}$) be the set of functions: $\mathbb{R} \to \mathbb{R}$. Define the operations

   $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$.

   Show that f($\mathbb{R}$) is a vector space under these operations.

2. Let $V = \{(x, y) : x \geq 0, y \geq 0\}$. Show that the set V fails to be a vector space under the standard operations on $\mathbb{R}^2$.

Linear combination of vectors

Definition: a linear combination of vectors $\vec{v_1}$, $\vec{v_2}$ .... $\vec{v_n}$ of a vector space ($\mathbb{R}$, V, +) is each sum of the form $\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \dots \alpha_n\vec{v_n}$ where $\alpha_1, \alpha_2, \dots \alpha_n$ are the scalars (the element of $\mathbb{R}$) which are called **coefficients of linear combination**.

Example 10.5

Show that the vector $\vec{w} = (14, 3)$ is a linear combination of vectors $\vec{u} = (4, 3)$ and $\vec{v} = (2, -1)$. Illustrate all the three vectors.

Solution

Let $\vec{w} = \alpha\vec{u} + \beta\vec{v}$

$(14, 3) = \alpha(4, 3) + \beta(2, -1)$

We form the system

$$\begin{cases} 14 = 4\alpha + 2\beta \\ 3 = 3\alpha - \beta \end{cases}$$

We solve for α and β

$$\begin{cases} 4\alpha + 2\beta = 14 \\ 3\alpha - \beta = 3 \end{cases}$$

$$\begin{cases} 4\alpha + 2\beta = 14 \\ 6\alpha - 2\beta = 6 \end{cases}$$

$$\begin{cases} 10\alpha = 20 \\ \beta = 3\alpha - 3 \end{cases}$$

$$\begin{cases} \alpha = 2 \\ \beta = 3 \end{cases}$$

$\alpha = 2$ and $\beta = 3$

Therefore, $\vec{w} = 2\vec{u} + 3\vec{v}$

$(14, 3) = 2(4, 3) + 3(2, -1)$.

The graph illustrating the vector $\vec{w} = (14, 3)$ expressed as a linear combination of the vectors $\vec{u} = (4, 3)$ and $\vec{v} = (2, -1)$ is given below.

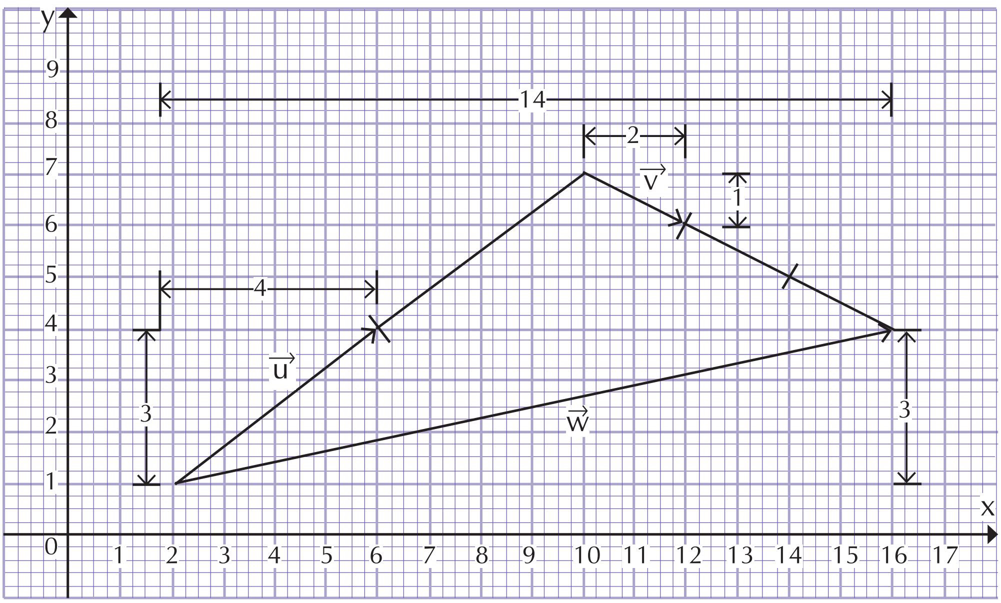


Fig. 10.4

Spanning vectors

Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a set of vectors in a vector space V. If every vector in V can be expressed in at least one way as a linear combination of the vector in S, then we say that the set S spans V, or that V is spanned by S.

Example 10.6

Let V be the vector space $\mathbb{R}^2$ and let $S = \{\vec{v}, \vec{w}\}$, where $\vec{v} = (2,1)$ and $\vec{w} = (1, 3)$. (a) Determine whether S spans $\mathbb{R}^2$.

(b) For $\vec{y} = (-2, 4)$, determine c_1 and c_2 such that $\vec{y} = c_1\vec{v} + c_2\vec{w}$

Solution

(a) We let $\vec{x} = (a, b)$ represent any vector in $\mathbb{R}^2$ (a and b are arbitrary real numbers). We must determine if there are constants c_1 and c_2 such that

$c_1\vec{v} + c_2\vec{w} = \vec{x}$

This leads to

$c_1(2, 1) + c_2(1, 3) = (a, b)$

From which we obtain the following system in 'unknowns' c_1 and c_2:

$$\begin{cases} 2c_1 + c_2 = a \\ c_1 + 3c_2 = b \end{cases}$$

We find the unique solution for c_1 and c_2 to be

$c_1 = \frac{3a - b}{5}$ and $c_2 = \frac{2b - a}{5}$

Thus $\{\vec{v}, \vec{w}\}$ spans $\mathbb{R}^2$.

(b) Since $a = -2$ and $b = 4$,

$c_1 = \frac{3a - b}{5} = \frac{-6 - 4}{5} = -\frac{10}{5} = -2$ and $c_2 = \frac{2b - a}{5} = \frac{8 - (-2)}{5} = \frac{10}{5} = 2$

Thus, $\vec{y} = -2\vec{v} + 2\vec{w}$.

Linear dependent vectors

We say that the vectors $\vec{v}_1, \vec{v}_2, \ldots\ldots, \vec{v}_n$ are linear dependent if we can find the scalars $\alpha_1, \alpha_2 \ldots \alpha_n$ which are not equal to zero such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots \alpha_n\vec{v}_n = \vec{0}$.

Example 10.7

Show that the vectors $\vec{u} = (1, -1)$ and $\vec{v} = (-2, 2)$ are linearly dependent.

Solution

$\forall, \alpha, \beta \in \mathbb{R}$: $\alpha\vec{u} + \beta\vec{v} = \vec{0}$ we have to find α and β.

$\alpha(1, -1) + \beta(-2, 2) = (0, 0)$, which gives us the system below

$$\begin{cases} 1\alpha - 2\beta = 0 \\ -1\alpha + 2\beta = 0 \end{cases} \quad \begin{cases} \alpha - 2\beta = 0 \\ -1\alpha + 2\beta = 0 \end{cases} \quad \begin{cases} \alpha = 2\beta \\ \alpha - 2\beta = 0 \end{cases}$$

Many values of α and β are different from zero ($\alpha = 2\beta$ for $\alpha \in \mathbb{R}$), hence $\vec{u}$ and $\vec{v}$ are linearly dependent.

Linear independent vectors

A set of vectors $\vec{v}_1, \vec{v}_2 \ldots\ldots . \vec{v}_n \in V$ is linearly independent if $\alpha\vec{v}_1 + \beta\vec{v}_2 + \ldots + \gamma\vec{v}_n = \vec{0}$ holds only if $\alpha = \beta = \ldots = \gamma = 0$; where $\alpha, \beta \ldots\ldots . \gamma$ are scalars i.e. $\alpha, \beta \ldots\ldots . \gamma \in \mathbb{R}$.

Otherwise, the set is linearly dependent.

Example 10.8

Show that the vectors $\vec{u} = (1, -1)$ and $\vec{v} = (-2, 3)$ are linearly independent.

Solution

$\forall, \alpha, \beta \in \mathbb{R}$: $\alpha\vec{u}, + \beta\vec{v} = \vec{0}$ we have to find α and β.

$\alpha(1, -1) + \beta(2, 3) = (0, 0)$, which gives us the system below:

$$\begin{cases} 1\alpha + 2\beta = 0 \\ -1\alpha + 3\beta = 0 \end{cases} \quad 0\alpha + 5\beta = 0;\ \beta = 0 \text{ and } \alpha = 0$$

Hence, the vectors $\vec{u}$ and $\vec{v}$ are linearly independent.

Application activity 10.2

1. Determine whether or not $\vec{u}$ and $\vec{v}$ are linearly dependent.
   (a) $\vec{u} = (3, 4)$, $\vec{v} = (1, 3)$
   (b) $\vec{u} = (2, 3)$, $\vec{v} = (6, 9)$
2. In $\mathbb{R}^2$, show that the vectors $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ are linearly independent.
3. Consider the set of vectors from $\mathbb{R}^2$, $S = \{\vec{v}, \vec{w}\}$, where $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.
   (a) Show that S spans $\mathbb{R}^2$.
   (b) Show that S is linearly independent.

Basis and dimension of a vector space

A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in a vector space V is called a basis for V if

(a) S spans V and

(b) S is linearly independent.

Example 10.9

Let $\vec{v}_1 = (3, 2)$ and $\vec{v}_2 = (1, 4)$. Show that the set $S = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.

Solution

To show that S spans $\mathbb{R}^2$, we must show that any vector $\vec{x} = (x_1, x_2)$ in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$$

This leads to the system of linear equations

$$\begin{cases} 3c_1 + c_2 = x_1 \\ 2c_1 + 4c_2 = x_2 \end{cases}$$

For S to span $\mathbb{R}^2$ we must show that this system has a solution for c_1 and c_2 for any possible choice of $\vec{x}$. Solving for c_1 and c_2 gives us:

$$\begin{cases} 12c_1 + 4c_2 = 4x_1 \\ -2c_1 - 4c_2 = -x_2 \end{cases} \quad \begin{cases} 10c_1 = 4x_1 - x_2 \\ c_2 = x_1 - 3c_1 \end{cases} \quad \begin{cases} c_1 = \dfrac{4x_1 - x_2}{10} \\ c_2 = x_1 - 3\left(\dfrac{4x_1 - x_2}{10}\right) \end{cases} \quad \begin{cases} c_1 = \dfrac{4x_1 - x_2}{10} \\ c_2 = \dfrac{10x_1 - 12x_1 + 3x_2}{10} \end{cases}$$

$$\begin{cases} c_1 = \dfrac{4x_1 - x_2}{10} \\ c_2 = \dfrac{-2x_1 + 3x_2}{10} \end{cases}$$ which are the values of c_1 and c_2

Thus, S spans $\mathbb{R}^2$.

To prove that S is linearly independent, we must show that the only solution of $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ is $c_1 = c_2 = \vec{0}$.

$$\begin{cases} 3c_1 + c_2 = 0 \\ 2c_1 + 4c_2 = 0 \end{cases}$$ and solving the system we obtain

$$\begin{cases} 12c_1 + 4c_2 = 0 \\ -2c_1 - 4c_2 = 0 \end{cases} \quad \begin{cases} 10c_1 = 0 \\ 4c_2 = -2c_1 \end{cases} \quad \begin{cases} c_1 = 0 \\ c_2 = \dfrac{1}{2}c_1 \end{cases} \quad \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \quad c_1 = c_2 = 0$$

Hence, the set $S = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$.

Note: The set $S = \{\vec{e}_1, \vec{e}_2\}$, where $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0, 1)$, is known as **the standard basis** for $\mathbb{R}^2$. Often $\vec{e}_1$ and $\vec{e}_2$ are replaced by the symbols $\vec{i}$ and $\vec{j}$, respectively. Also, for $\vec{x} = (x_1, x_2)$ we have $\vec{x} = (x_1, x_2) = x_1\vec{e}_1 + x_2\vec{e}_2$.

Dimension of vector space

The dimension of a non-zero vector space V is the number of vectors in a basis for V. Often we write Dim (V) for the dimension of V.

For $\mathbb{R}^n$, one basis is the standard basis, and it has n vectors. Thus, the dimension of $\mathbb{R}^n$ is n.

Example 10.10

Given that $V = \{(3, 2), (1, 3)\}$

(a) show that V is a basis of $\mathbb{R}^2$

(b) determine the coordinate of (1, 0) and (0, 1) in that base

(c) find the dimension of that base.

Solution

(a) Show that the vectors (3, 2), (1, 3) are linearly independent:

$\forall(3, 2), (1, 3) \in \mathbb{R}^2$, and $\forall\ \alpha, \beta \in \mathbb{R}$ such that $\alpha(3, 2), + \beta(1, 3) = 0$

We obtain the system below

$$\begin{cases}3\alpha + \beta = 0\\2\alpha + 3\beta = 0\end{cases} \Leftrightarrow \begin{cases}6\alpha + 2\beta = 0\\-6\alpha - 9\beta = 0\end{cases} \Leftrightarrow \begin{cases}3\alpha = -\beta\\-7\beta = 0\end{cases} \Leftrightarrow \begin{cases}\beta = 0\\\alpha = 0\end{cases} \Leftrightarrow \beta = 0 \text{ and } \alpha = 0$$

So those vectors are linearly independent.

Show if $V = \{(3, 2), (1, 3)\}$ are generators of V.

$\forall(3, 2), (1, 3) \in \mathbb{R}^2$, and $\exists\ \alpha, \beta \in \mathbb{R}$

$\forall(x, y) \in \mathbb{R}^2;\ (x, y) = \alpha(3, 2) + \beta(1, 3)$

$(x, y) = (3\alpha, 2\alpha) + (\beta, 3\beta)$

We obtain the system below

$$\begin{cases}x = 3\alpha + \beta\\y = 2\alpha + 3\beta\end{cases}$$

$$\begin{cases}6\alpha + 2\beta = 2x\\-6\alpha - 9\beta = -3y\end{cases} \Leftrightarrow \begin{cases}-7\beta = 2x - 3y\\3\alpha = x - \beta\end{cases} \Leftrightarrow \begin{cases}\beta = -\dfrac{2x - 3y}{7}\\\alpha = \dfrac{x - \beta}{3}\end{cases}$$

$$\begin{cases}\beta = \dfrac{3y - 2x}{7}\\\alpha = \dfrac{x - \left(\dfrac{3y - 2x}{7}\right)}{3}\end{cases} \Leftrightarrow \begin{cases}\beta = \dfrac{3y - 2x}{7}\\\alpha = \dfrac{9x - 3y}{21}\end{cases} \Leftrightarrow \begin{cases}\beta = \dfrac{3y - 2x}{7}\\\alpha = \dfrac{3x - y}{7}\end{cases}$$

We see that α and β exist and the vectors are generators of V.

Finally, V is a base of $\mathbb{R}^2$.

(b) To find the coordinates of (1, 0) and (0, 1) we proceed as follows:

Let (x, y) be the coordinates of (1, 0) in V such that $(1, 0) = x\,(3, 2) + y\,(1, 3)$

Then we have to find x and y.

$$\begin{cases}3x + y = 1\\2x + 3y = 0\end{cases} \Leftrightarrow \begin{cases}6x + 2y = 2\\-6x - 9y = 0\end{cases} \Leftrightarrow \begin{cases}-7y = 2\\3x = 1 - y\end{cases}$$

$$\begin{cases}y = -\dfrac{2}{7}\\x = \dfrac{1 - y}{3}\end{cases} \Leftrightarrow \begin{cases}y = -\dfrac{2}{7}\\x = \dfrac{1 + \dfrac{2}{7}}{3}\end{cases} \Leftrightarrow \begin{cases}y = -\dfrac{2}{7}\\x = \dfrac{3}{7}\end{cases}$$

So the coordinates of (1, 0) in that base are $(\frac{3}{7}, -\frac{2}{7})$.

In the same way, we can easily find the coordinates of (0,1) in that base

Let (x, y) be the coordinates of (0, 1) in V such that (0, 1) = x (3, 2) + y (1, 3)

Then we have to find x and y.

$$\begin{cases} 3x + y = 0 \\ 2x + 3y = 1 \end{cases} \Leftrightarrow \begin{cases} 6x + 2y = 0 \\ -6x - 9y = -3 \end{cases} \Leftrightarrow \begin{cases} -7y = -3 \\ 3x = -y \end{cases}$$

$$\begin{cases} y = \frac{3}{7} \\ x = \frac{-y}{3} \end{cases} \Leftrightarrow \begin{cases} y = \frac{3}{7} \\ x = \frac{-\frac{3}{7}}{3} \end{cases} \Leftrightarrow \begin{cases} y = \frac{3}{7} \\ x = -\frac{1}{7} \end{cases}$$

The coordinates of (0, 1) in basis {(3, 2), (1, 3)} are $(-\frac{1}{7}, \frac{3}{7})$.

(c) The dimension of V is equal to 2 because V has two vectors which are linearly independent (dim (V) = 2).

10.3 Euclidian vector space

Activity 10.1

Carry out research to determine the similarities between vector spaces and Euclidian spaces.

A Euclidean space is a real vector space L in which to any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ and any scalar α, there corresponds a real number $(\vec{u}, \vec{v})$ such that the following conditions are satisfied:

(1) $(\vec{u} + \vec{v}, \vec{w}) = (\vec{u}, \vec{w}) + (\vec{v}, \vec{w})$

(2) $(\alpha\vec{u}, \vec{v}) = \alpha(\vec{u}, \vec{v})$

(3) $(\vec{u}, \vec{v}) = (\vec{v}, \vec{u})$

(4) $(\vec{u}, \vec{u}) > 0$ for $\vec{u} \neq 0$

The expression $(\vec{u}, \vec{u})$ is frequently denoted by $(\vec{u})$; it is called the **scalar square** of the vector $\vec{u}$. Thus property (4) implies that the quadratic form corresponding to the bilinear form $(\vec{u}, \vec{u})$ is positive definite.); it is called the scalar square of the vector $\vec{u}$.

The magnitude of vector $\overrightarrow{AB}$

If $A(x_1, y_1)$ and $B(x_2, y_2)$ then we have the vector $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$.

The Magnitude of the vector $\overrightarrow{AB}$ is given by $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 10.11

If A = (3, 1) and B = (6, –3) find

a) $\overrightarrow{AB}$

b) $|\overrightarrow{AB}|$

Solution

a) $\overrightarrow{AB} = \binom{6-3}{-3-1} = \binom{3}{-4}$

b) $|\overrightarrow{AB}| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5$

Unit vector

A unit vector has a length of one unit.

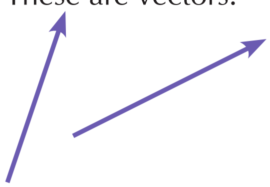
The unit vector in the direction of the vector a is the vector $\vec{a} = (\frac{1}{|\vec{a}|})\vec{a}$

The unit vector in the direction of the positive x and y axes are denoted by $\vec{i}$ and $\vec{j}$ respectively.

Thus $\vec{i}\binom{1}{0}$ and $\vec{j}\binom{0}{1}$.

Dot product and properties

These are vectors:



They can be **multiplied** using the "**dot product**". Let us use Figure 10.5 to show how we can calculate the dot product of two vectors:

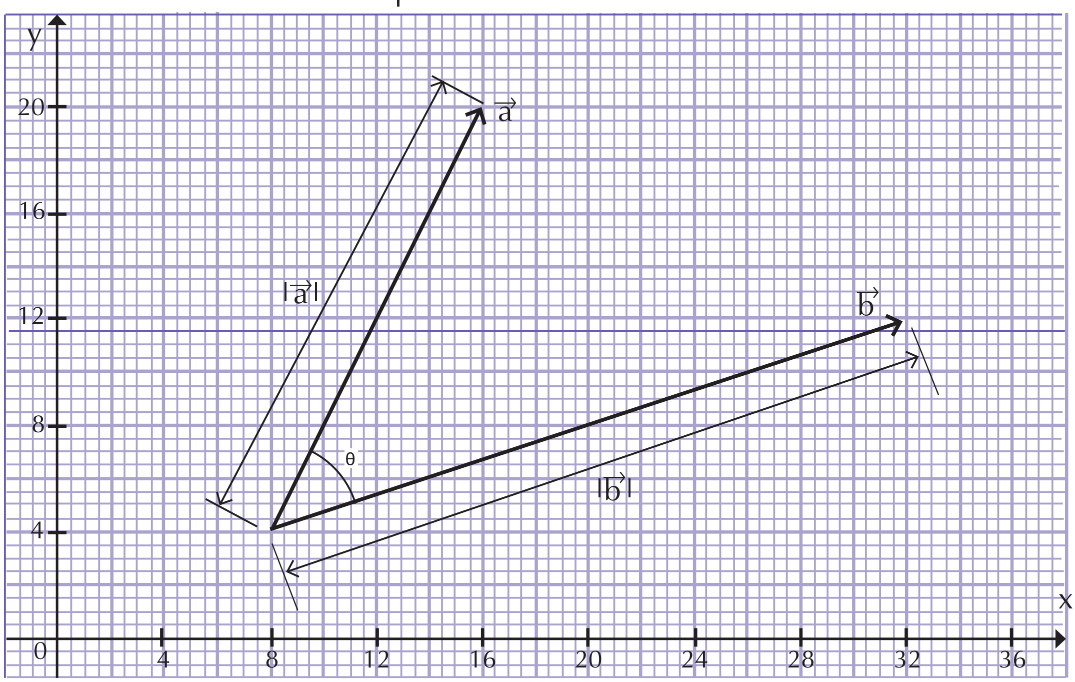


Fig. 10.5

$\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos(\theta)$

Where:

$|\vec{a}|$ is the magnitude (length) of vector $\vec{a}$

$|\vec{b}|$ is the magnitude (length) of vector $\vec{b}$

θ is the angle between $\vec{a}$ and $\vec{b}$.

So we multiply the length of $\vec{a}$ times the length of $\vec{b}$, then multiply by the cosine of the angle between $\vec{a}$ and $\vec{b}$.

Or, we can calculate Using the example illustrated Figure 10.6.

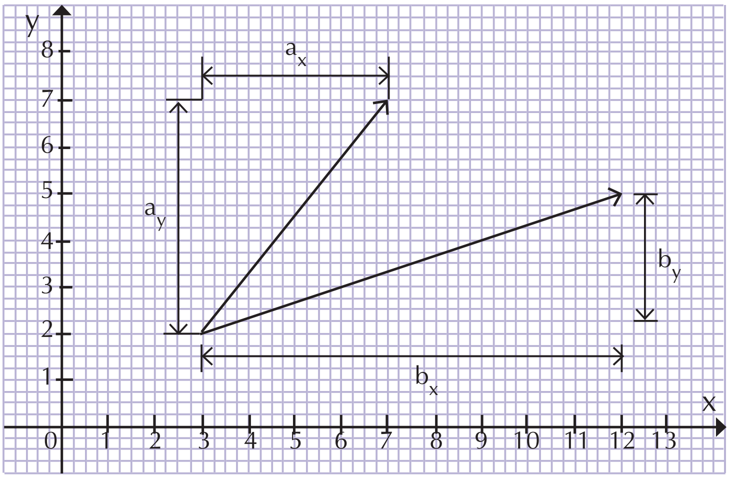


Fig. 10.6

$\vec{a} \bullet \vec{b} = a_x \times b_x + a_y \times b_y$

Where:

a_x is the x component of the vector $\vec{a}$

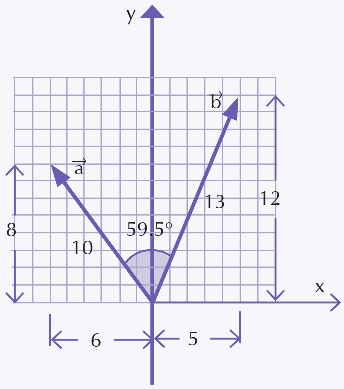
b_x is the x component of the vector $\vec{b}$

a_y is the y component of the vector $\vec{a}$

b_y is the y component of the vector $\vec{b}$

Example 10.12

Calculate the dot product of vectors $\vec{a}$ and $\vec{b}$.



$\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos(\theta)$

$\vec{a} \cdot \vec{b} = 10 \times 13 \times \cos(59.5°)$

$\vec{a} \cdot \vec{b} = 10 \times 13 \times 0.5075...$

$\vec{a} \cdot \vec{b} = 65.98... = 66$ (rounded off)

Alternatively,

$\vec{a} \cdot \vec{b} = a_x \times b_x + a_y \times b_y$

$\vec{a} \cdot \vec{b} = -6 \times 5 + 8 \times 12$

$\vec{a} \cdot \vec{b} = -30 + 96$

$\vec{a} \cdot \vec{b} = 66$

Both methods give the same result (after rounding off)

Also note that we used **minus 6** for a_x (it is heading in the negative x-direction)

Consider two vectors of any vector space.

Let $\vec{a} = (x, y)$ and $\vec{v} = (x', y')$ be the two vectors of V. Then the scalar product of $\vec{u}$ and $\vec{v}$ is a scalar $\vec{u} \cdot \vec{v} = xx' + yy'$

The scalar product is often called the dot product since a 'dot' is used to denote it.

Example 10.13

Let $\vec{u} = (3, 4)$ and $\vec{v} = (-1, 4)$. Find the dot product of $\vec{u}$ and $\vec{v}$.

Solution

The dot product of $\vec{u}$ and $\vec{v}$ is

$\vec{u} \cdot \vec{v} = (3, 4)\ (-1, 4) = 3(-1) + 4(4) = -3 + 16 = 13$

Example 10.14

Calculate the dot product.

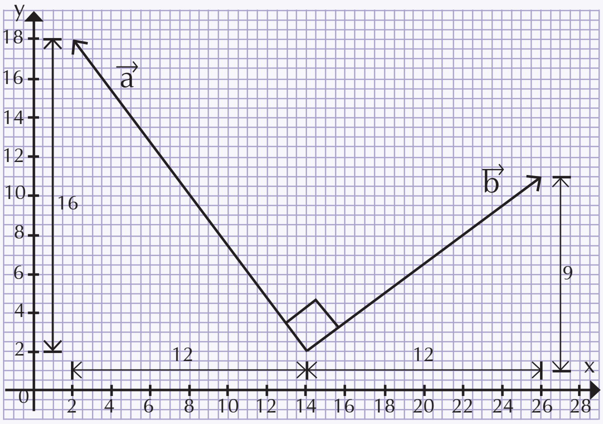


Fig. 10.7

Solution

$\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos(\theta)$

$\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos(90°)$

$\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times 0$

$\vec{a} \cdot \vec{b} = 0$

$\vec{a} \cdot \vec{b} = a_x \times b_x + a_y \times b_y$

$\vec{a} \cdot \vec{b} = -12 \times 12 + 16 \times 9$

$\vec{a} \cdot \vec{b} = -144 + 144$

$\vec{a} \cdot \vec{b} = 0$

Properties of the scalar product

Activity 10.2

Research on and discuss following properties of the scalar product of two vectors.

$\vec{u}.\vec{v} = \vec{v}.\vec{u}$

$\vec{u}.\vec{u} = |\vec{u}|^2$

$\vec{u}.(\vec{v} + \vec{w}) = (\vec{u}.\vec{v}) + (\vec{u}.\vec{w})$

$(\vec{u} + \vec{v}).(\vec{u} - \vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$

Angle between two vectors

If $\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}| \cos\theta \Leftrightarrow \cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Here, $\cos\theta$ means the cosine of the angle θ between the vectors, $\vec{a}$ and $\vec{b}$ where $0° \leq \theta \leq 180°$.

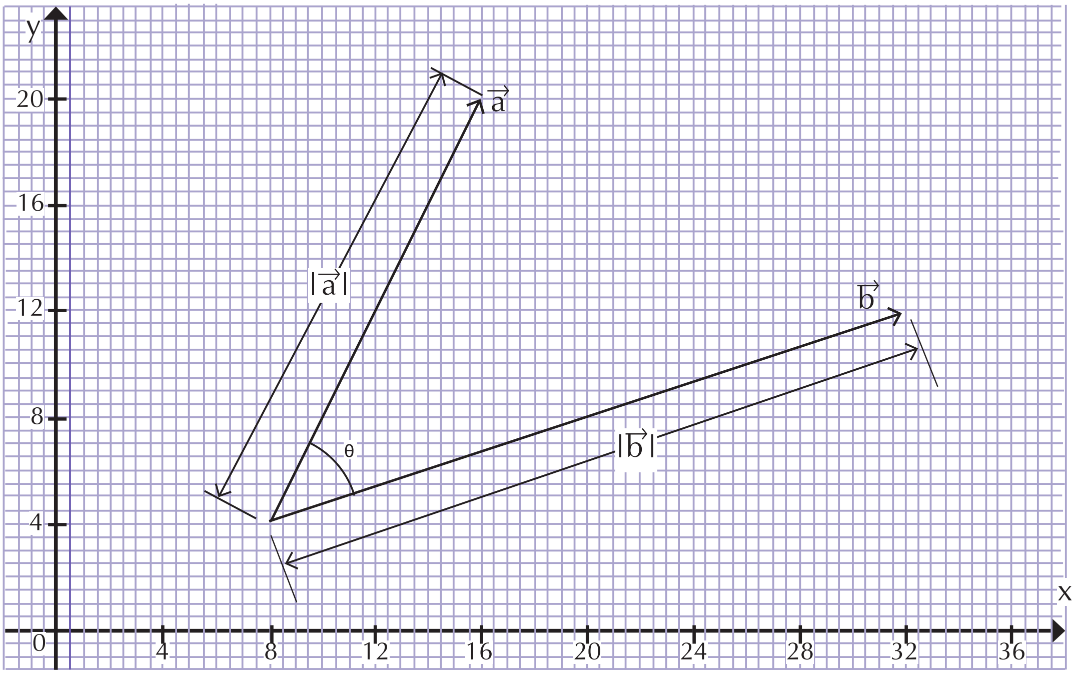


Fig. 10.8

Note: Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular (orthogonal) if and only if their dot product is zero i.e. $\vec{a} \bullet \vec{b} = 0$.

Example 10.15

Determine the angle between the vectors $\vec{u} = (4, 3)$ and $\vec{v} = (2, 6)$.

Solution

$|\vec{u}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

$|\vec{v}| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$

$\vec{u} \cdot \vec{v} = 4(2) + 3(6) = 8 + 18 = 26$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{1}{5 \times 2\sqrt{10}} = \frac{13}{5\sqrt{10}} = 0.822$$

Using a calculator, we find that $\theta = 34.7°$ (or 0.605 radians).

Example 10.16

Determine the angle between the two vectors $\vec{a} = (2, 1)$ and $\vec{b} = (-2, 4)$.

Solution

Here, we have $\vec{a} \cdot \vec{b} = 2(-2) + 1(4) = -4 + 4 = 0$. Thus, $\vec{a}$ and $\vec{b}$ are mutually perpendicular ($\theta = 90°$).

Activity 10.3

Solve the following:

1. Vector $\vec{a}$ has magnitude 3, vector $\vec{b}$ has magnitude 4 and the angle between $\vec{a}$ and $\vec{b}$ is 60°.

   What is the value of $\vec{a}.\vec{b}$?

2. Vector $\vec{a}$ has magnitude $3\sqrt{2}$, vector $\vec{b}$ has magnitude 5 and the angle between $\vec{a}$ and $\vec{b}$ is 135°.

   What is the value of $\vec{a}\,.\,\vec{b}$?

3. In Figure 10.9, what is the value of $\vec{a}\,.\,\vec{b}$?

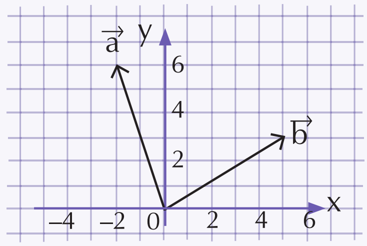


Fig. 10.9

4. Use the dot product to find the size of angle θ in Figure 10.10.

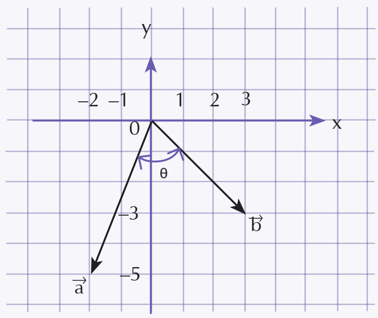


Fig. 10.10

5. Find the angle θ in Figure 10.11.

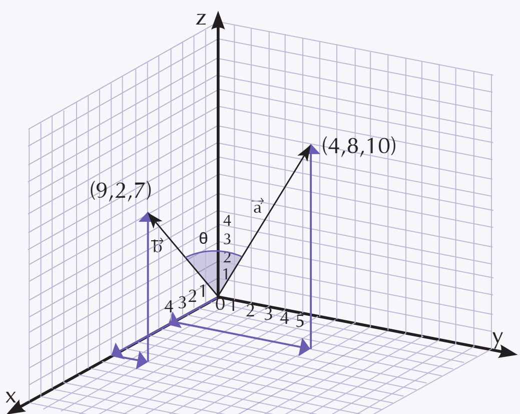


Fig. 10.11

Summary

1. A quantity which has both **magnitude** (size) and **direction** is called a **vector**.
2. Vectors are added by **the triangle law** or end-on rule or **parallelogram law** of addition.
3. Vector addition is commutative i.e vectors can be added in any order.
4. The vector $\vec{a} - \vec{b}$ is equal to $\vec{a} + (-\vec{b})$. Thus $\vec{a} - \vec{b}$ can be found by adding vector $-\vec{b}$ to vector $\vec{a}$.
5. When a vector is multiplied by a scalar its magnitude changes. If it is multiplied by a positive number the direction remains the same. However, if it is multiplied by a negative number, the direction of the vector reverses.
6. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots\ldots, \vec{v}_n$ of a vector space $(\mathbb{R}, V, +)$ is each sum of the form $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots\alpha_n\vec{v}_n$ where $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ are the scalars (the element of $\mathbb{R}$) which are called **coefficients of linear combination.**
7. A set of vectors $S = \{\vec{v}_1, \vec{v}_2 \ldots\ldots, \vec{v}_n\}$ in a vector space V is called a basis for V if
   a. S spans V and
   b. S is linearly independent.
8. A **unit vector** has a length of one unit.
9. If $\vec{u}$ and $\vec{v}$ are two non-zero vectors, then $\vec{u}.\vec{v} = |\vec{u}||\vec{v}| \cos\theta \Leftrightarrow \cos\theta = \frac{\vec{u}.\vec{v}}{|\vec{u}||\vec{v}|}$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Topic area: Linear algebra

Sub-topic area: Linear transformations in 2D

Unit 11

Concepts and operations on linear transformations in 2D

Key unit competence

Determine whether a transformation of IR^2 is linear or not. Perform operations on linear transformations.

11.0 Introductory activity

Let consider a triangle with vertices $A(2,1), B(3,3), C(-1,3)$ and another with vertices $A'(0,4), B'(1,6), C'(-3,6)$

1. Sketch the triangle ABC and $A'B'C'$ in Cartesian plane.
2. With an arrow join vertex A and A', B and B' , C and C'.
3. What is the relation between triangle ABC and triangle $A'B'C'$?

11.1 Linear transformation in 2D

Activity 11.1

What is transformation as is used in mathematics? What is linear transformation? Research on the different types of 2D linear transformations. Discuss your findings with the rest of the class.

Definitions

A **linear transformation** T from a vector space V to a vector space W is a function T: V $\rightarrow$ W that satisfies the following two conditions:

(1) $\forall \vec{u}, \vec{v} \in V$: $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $\forall \vec{u} \in V$, and scalar α : $T(\alpha\vec{u}) = \alpha T(\vec{u})$

Example 11.1

Show that T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x,y) = (x, x + y)$ is a linear transformation.

Solution

We verify the two conditions of the definition.

For the first condition, we let (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ and compute

$T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, x_1 + x_2 + y_1 + y_2)$

$= (x_1, x_1 + y_1) + (x_2, x_2 + y_2) = T(x_1, y_1) + T(x_2, y_2)$

This proves the first condition.

For the second condition, we let $\alpha \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ and we compute

$T(\alpha(x,y)) = T(\alpha x, \alpha y) = (\alpha x, \alpha x + \alpha y) = \alpha(x, x + y) = \alpha T(x,y)$

Hence T is a linear transformation.

Example 11.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x, 1)$. Show that T is not linear.

Solution

We show the first condition of the definition is violated. Indeed, for any two vectors (x_1, y_1) and (x_2, y_2) we have.

$T((x_1 y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, 1)$

$\neq (x_1, 1) + (x_2, 1) = T(x_1, y_1) + T(x_2, y_2)$

Hence the given transformation is not linear.

Application activity 11.1

1. In pairs, determine whether or not the following are linear transformations
   a) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x, y) = (y, 0)$
   b) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x, y) = (x^2, 0)$
   c) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x, y) = (x, 2x - y)$
   d) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x, y) = (1, xy)$
2. Define $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x,y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} . \begin{pmatrix} x \\ y \end{pmatrix}$ State whether or not f is a linear transformation.

Properties of linear transformations

If $T: V \to W$ is a linear transformation then

$T(0) = 0$

$T(-u) = -T(u)$

$T(u - v) = T(u) - T(v)$

$T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots \alpha_n T(u_n)$.

11.2 Geometric transformations in 2D

Activity 11.2

Research on the meaning of geometric transformations. How many types can you list, with examples? Discuss your findings in class.

A geometric transformation is an operation that modifies the position, size, shape and orientation of the geometric object with respect to its current state and position.

We say that the point $P = \begin{pmatrix} x \\ y \end{pmatrix}$ has an image $P' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ under the transformation.

Every linear transformation in 2D can be expressed using a 2x2 matrix and it is called the transformation matrix.

Given the transformation $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x,y) = (ax + by, cx + dy)$ we can write it as

$\begin{pmatrix} x' \\ y' \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ or writing in matrix form $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Reflection

Generally, a reflection is a transformation representing a flip of a figure. Figures may be reflected in a point, a line, or a plane. When reflecting a figure in a line or in a point, the image is congruent to the preimage.

A reflection maps every point of a figure to an image across a line of symmetry using a reflection matrix.

Use the following rule to find the reflected image across a line of symmetry using a reflection matrix.

For a reflection over the:	x – axis	y – axis	line y = x
Multiply the vertex on the left by	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 11.3

Find the coordinates of the vertices of the image of pentagon *ABCDE* with *A*(2, 4), *B*(4, 3), *C*(4, 0), *D*(2, –1), and *E*(0, 2) after a reflection across the *y*-axis.

Write the ordered pairs as a vertex matrix.

$$\begin{bmatrix} 2 & 4 & 4 & 2 & 0 \\ 4 & 3 & 0 & -1 & 2 \end{bmatrix}$$

To reflect the pentagon $ABCDE$ across the y-axis, multiply the vertex matrix by the reflection matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 4 & 2 & 0 \\ 4 & 3 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -4 & -2 & 0 \\ 4 & 3 & 0 & -1 & 2 \end{bmatrix}$$

Therefore, the coordinates of the vertices of the image of pentagon $ABCDE$ are $A'(-2, 4)$, $B'(-4, 3)$, $C'(-4, 0)$, $D'(-2, -1)$, and $E'(0, 2)$.

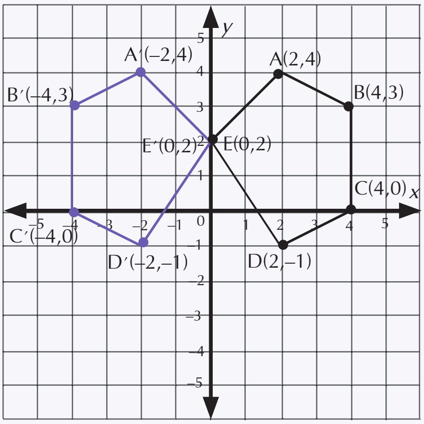


Fig 11.1

Notice that, both figures have the same size and shape.

Reflection (symmetry) about x-axis

$f(\overrightarrow{e_1}) = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$f(\overrightarrow{e_2}) = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

The standard transformation matrix is $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

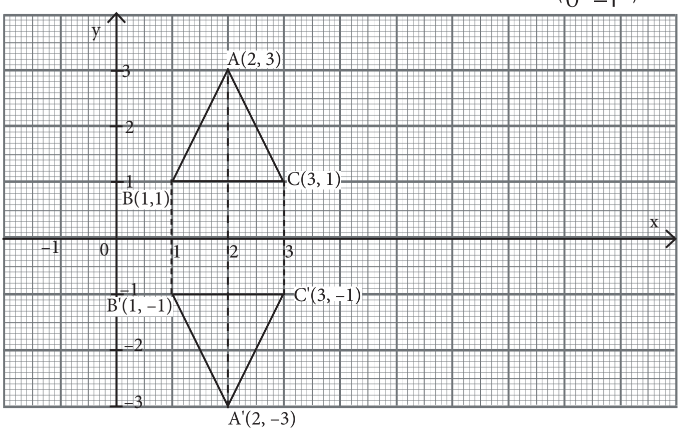


Fig 11.2

Example 11.4

Find the image of the point (6, – 3) reflected in the x-axis.

Solution

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 6-0 \\ 0+3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

Thus, the image of (6, –3) reflected in the x-axis is (6, 3).

Generally, if f is the reflection in the x-axis in $\mathbb{R}^2$, then f(x, y) = (x, – y).

Reflection (symmetry) about y-axis

$f(\overrightarrow{e_1}) = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and

$f(\overrightarrow{e_2}) = f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The standard transformation matrix is $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

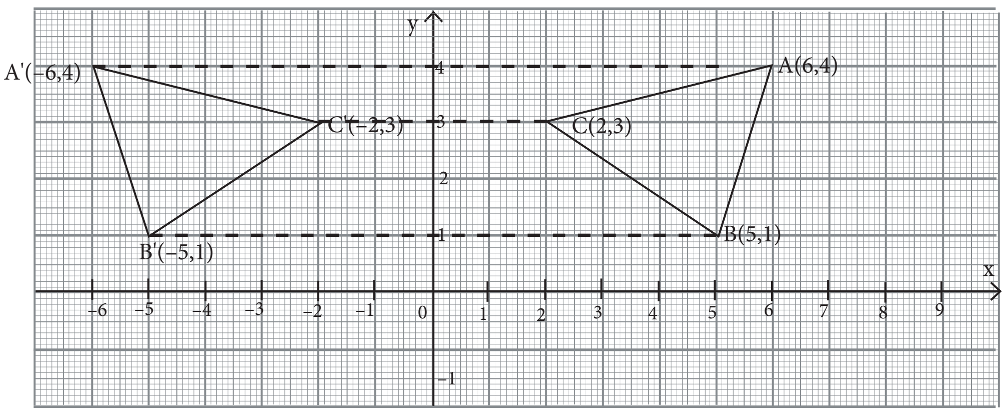


Fig 11.3

Example 11.5

Find the image of the point (1,2) reflected in the y-axis.

Solution

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+0 \\ 0+2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Thus, the image of (1,2) reflected in the y-axis is (–1,2).

Generally, if f is the reflection in the y-axis in $\mathbb{R}^2$, then f(x,y) = (–x,y).

Reflection (symmetry) about the line y = x

The standard transformation matrix is $M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

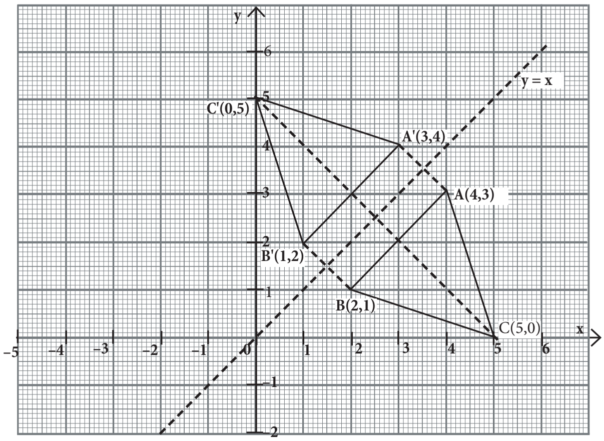


Fig 11.4

Example 11.6

Use the standard transformation matrix in the line y = xy to compute the image of the point (–3, 4).

Solution

$f(-3, 4) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0+4 \\ -3+0 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

Generally, if f is the reflection in the line y = xy in $\mathbb{R}^2$, then f(x,y) = (y,x).

Central symmetry

We call the central symmetry of the point A about the origin O if the distance AO is equal to AO′ and we say that the image of the point A about origin O is A′. $f(\overrightarrow{e_1}) = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $f(e_2) = f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The standard tranformation matrix is $+[f(\overrightarrow{e_2}), f(e_2)] = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

Generally in $\mathbb{R}^2$ the central symmetry in the origin is f(x,y) = (–x, –y).

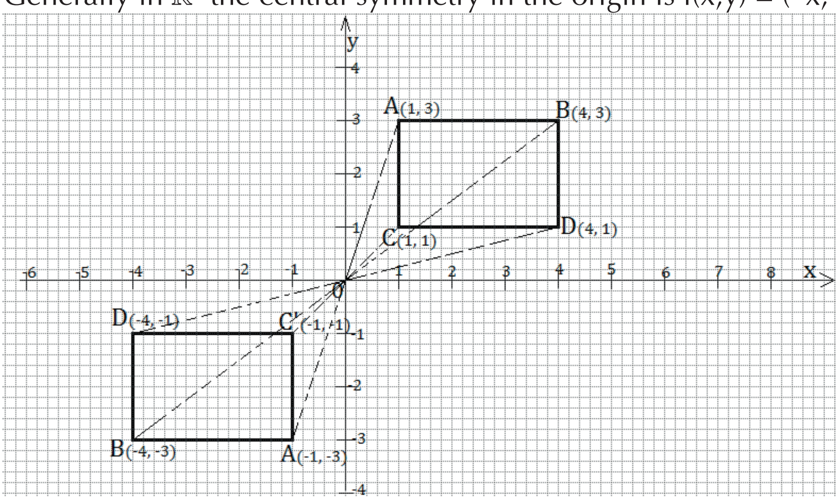


Fig 11.5

Identical transformation

In identical transformation, the image of the point remains the original one. The standard transformation matrix is $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Rotation

A rotation is a **transformation** in a plane that turns every point of a preimage through a specified angle and direction about a fixed point. The fixed point is called the **center of rotation**. The amount of rotation is called the **angle of rotation** and it is measured in degrees.

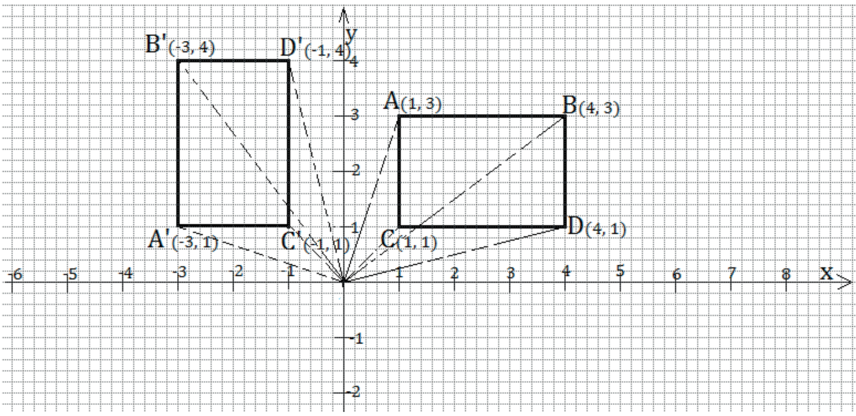


Fig 11.6

A rotation maps every point of a preimage to an image rotated about a centre point, usually the origin, using a rotation matrix.

Use the following rules to rotate the figure for a specified rotation. To rotate counterclockwise about the origin, multiply the vertex matrix by the given matrix.

Angle of rotation	90°	180°	270°
Rotation matrix (Multiply on the left)	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Example 11.7

Find the coordinates of the vertices of the image ΔXYZ with $X(1, 2)$, $Y(3, 5)$, and $Z(-3, 4)$ after it is rotated 180° counterclockwise about the origin.

Write the ordered pairs as a vertex matrix.

$$\begin{bmatrix} 1 & 3 & -3 \\ 2 & 5 & 4 \end{bmatrix}$$

To rotate the Δ XYZ through180° counterclockwise about the origin, multiply the vertex matrix by the rotation matrix, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -3 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 3 \\ -2 & -5 & -4 \end{bmatrix}$$

Therefore, the coordinates of the vertices of ΔX′Y′Z′ are X′(–1,,–2), Y′(–3,–5), and Z′(3,–4).

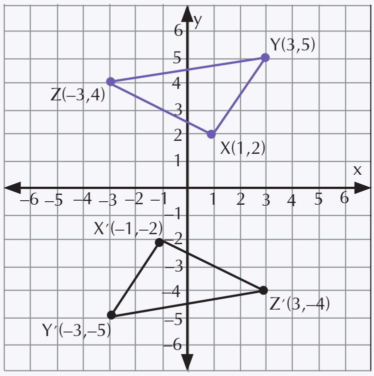


Fig 11.7

Notice that the image (ΔX′Y′Z′) is congruent to the preimage (ΔXYZ). Both figures have the same size and same shape.

Rotation about origin with angle θ

If a point (x, y) in the plane is rotated counterclockwise about the origin through an angle θ to obtain a new point (x′, y′), then

$x' = x \cos\theta - y \sin\theta$

$y' = x \sin\theta + y \cos\theta$

This can be written as $\begin{pmatrix} x' \\ y' \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Thus, the standard transformation matrix for the rotation about the origin with an angle θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Example 11.8

Find the new coordinates of the point (2,4) after its rotation through an angle of θ = 60°.

Solution

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 - 2\sqrt{3} \\ \sqrt{3} + 2 \end{pmatrix}$$

Orthogonal and orthonormal transformation

A set of vectors is said to be orthogonal if all pairs of vectors in the set are perpendicular.

A set of vectors is said to be orthonormal if it is an orthogonal set and all the vectors have unit length.

Orthogonal projection of a vector

Suppose that $\vec{v} \in \mathbb{R}^2$ is a vector. Then, for $\vec{u} \in \mathbb{R}^2$ the projection of $\vec{u}$ on $\vec{v}$ is defined as $Proj_v \vec{u} = \frac{\vec{u}.\vec{v}}{|\vec{v}|^2}\vec{v}$.

Example 11.9

Let $\vec{u} = (2, 5)$ and $\vec{v} = (4, 3)$. Find the projection of $Proj_v\vec{u}$.

Solution

$$Proj_v\vec{u} = \frac{\vec{u}.\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{(4,3).(2,5)}{\sqrt{4^2+3^2}}(4,3) = \frac{8+15}{\sqrt{25}}(4,3) = \frac{23}{5}(4,3) = \left(\frac{92}{5}, \frac{69}{5}\right)$$

Example 11.10

Let f be the projection on to the vector $\vec{v} = (-1, 2)$ in $\mathbb{R}^2 : f(\vec{u}) = Proj_v\vec{u}$.

a) Find the standard matrix.

b) Compute f(3, 1).

Solution

(a) $f(x,y) = Proj_v(x,y) = \frac{v.(x, y)}{|v|^2} v = \frac{(-1, 2).(x,y)}{\sqrt{(-1)^2+2^2}}(-1,2) = \frac{-x+2y}{\sqrt{5}}(-1,2)$

$$= \left(\frac{x-2y}{\sqrt{5}}, \frac{-2x+4y}{\sqrt{5}}\right)$$

Then, we know the standard basis of $\mathbb{R}^2$ are $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$ and then

$$f(e_1) = f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}}\end{pmatrix} \text{ and } f(e_2) = f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{-2}{\sqrt{5}}\\ \frac{4}{\sqrt{5}}\end{pmatrix}$$

Thus, the standard matrix is $M = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}}\end{pmatrix}$

(b) $f(3,1) = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{6}{\sqrt{5}} + \frac{4}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

Thus, $f(3,1) = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

11.3 Kernel and range

Kernel and image of linear transformation

The kernel of a linear transformation

Let $f: V \to W$ be a linear transformation of a vector space V into the vector space W, the kernel of f denoted by ker f is the set of all vectors of V which have zero vectors of W as their image.

$$Ker\ f = \left\{\vec{u} \in V : f\left(\vec{v}\right) = \vec{0} \in W\right\}$$

Example 11.11

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$: be an endomorphism of $\mathbb{R}^2$ such that $f(x,y) = (2x - y, -2x + y)$. Find

a) ker f

b) dim ker f.

Solution

$\ker f = \{\vec{v}\ \mathbb{R}^2 : f(\vec{v}) = 0 \in \mathbb{R}^2\} = \{(x, y) \in \mathbb{R}^2 : (2x - y, -2x + y) = (0, 0)\}$

$= \{(x,y)\ \mathbb{R}^2 : \begin{cases} 2x - y = 0 \\ -2x + y = 0 \end{cases} = \{(x, y) \in \mathbb{R}^2 : 2x - y = 0\} = \{(x, y) \in \mathbb{R}^2 : y = 2x\}$

$= \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\} = (x(1, 2) \in \mathbb{R}^2 : x \in \mathbb{R}\}$.

$\ker f = \{(x(1, 2) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Thus, $\dim \ker f = 1$

The range (image) of a linear transformation

Let $f: V \to W$ be a linear transformation of a vector space V into the vector space W, the range of f denoted by im f is the set of all vectors images in f

$\text{Im } f = \{f(\vec{v}\ ) \in W : \vec{v}\ \in V\}$

Example 11.12

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism of $\mathbb{R}^2$ such that $f(x, y) = (x - y, y)$; find

(a) Im f

(b) Dim Im f.

Let f: $\mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism of $\mathbb{R}^2$ such that f(x, y) = (x – y, y); find

(a) Im f

(b) Dim Im f.

Solution

(a) Now, let (a, b) $\in$ Im (f) be given. There exists a vector (x, y) $\in \mathbb{R}^2$ such that f(x, y) = (x – y, y) = (a, b). This yields the following system

$\begin{cases} x - y = a \\ y = b \end{cases}$ $\begin{cases} x = a + y \\ y = b \end{cases}$ $\begin{cases} x = a + b \\ y = b \end{cases}$ and (a, b) leads to the solution given by

(a,b) = (a + b,b) = (a, 0) + (b,b) = a(1,0) + b(1, 1); a, b $\in \mathbb{R}$

Hence, Im (F) = span {(1, 0), (1, 1)}

(b) Dim Im f = 2

Theorem: Let f: V $\to$ W be a linear transformation. Then

(a) ker(f) is a subspace of V.

(b) Im (f) is a subspace of W.

Nullity and rank

The dimension of the kernel is called the **nullity** of f (denoted nullity f) and the dimension of the range of f is called the **rank** of f (denoted rank f).

Example 11.13

Let f: $\mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism of $\mathbb{R}^2$ such that f(x,y) = (x,x + y); find

(a) nullity f

(b) rank f.

Solution

(a) ker f = {v $\in \mathbb{R}^2$: f($\vec{v}$) = 0 $\in \mathbb{R}^2$} = {(x, y) $\in \mathbb{R}^2$: (x, x + y) = (0, 0)}

={(x, y) $\in \mathbb{R}^2$: $\begin{cases} x = 0 \\ x + y = 0 \end{cases}$ } = {(x, y) $\in \mathbb{R}^2$: $\begin{cases} x = 0 \\ y = 0 \end{cases}$

= (x, y) $\in \mathbb{R}^2$ = (0,0)

Ker f = {(0, 0) $\in \mathbb{R}^2$}. Thus, dim ker f = 0

Nullity f = dim ker f = 0

(b) Now, let (a, b) $\in$ Im be given. There exists a vector (x, y) $\in \mathbb{R}^2$ such that f(x, y) = (x,x + y) = (a, b). This yields the following system:

$\begin{cases} x = a \\ x + y = b \end{cases}$ $\begin{cases} x = a \\ y = -x + b \end{cases}$ $\begin{cases} x = a \\ y = -a + b \end{cases}$ and (a,b) leads to the solution given by

$(a, b) = (a, -a + b) = (a,-a) + (0,b) = a (1,-1) + b(0,1); a,b \in \mathbb{R}$

Hence, Im (f) = span {(1, –1), (0,1)}

Dim im f = 2

Rank f = Dim im f = 2

Application activity 11.2

Determine Ker(L) for the following linear transformations. Also, find Dim[Ker(L)].

$L: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x) = \begin{bmatrix} 3x_1 + 2x_2 \\ -x_1 + x_2 \end{bmatrix}$

11.4 Operations of linear transformation

Activity 11.3

Carry out research to find out what is linear transformation. How do we sum up two linear transformations and how do we compose them?

Addition

$f: V \to W$ and $g: V \to W$

$f + g = V \to W: (f + g)(a) = f(a) + g(a)$. The sum of two linear transformations is a linear transformation.

Composition of two linear transformations

Let $f: V \to W$ and $g: V \to W$ then the composite of f and g, $(g \circ f)$ is a linear transformation

Proof

1. $(g \circ f)(a+b) = g\left(f(a+b)\right) = g\left(f(a) + f(b)\right)$

$= g\left(f(a)\right) + g\left(f(b)\right)$

$= (g \circ f)(a) + (g \circ f)(b)$

2. $(g \circ f)(\alpha a) = g(f(\alpha a)) = g(\alpha f(a)) = \alpha g(f(a)) = \alpha (g \circ f)(a)$.

Therefore, the composition of two linear transformations is also a linear transformation.

One-to-one linear transformation

A linear transformation f is said to be a one-to one if for each element in the range there is a unique element in the domain which maps to it.

Onto linear transformation

A linear transformation f is said to be onto if for every element in the range space there exists an element in the domain that maps to it.

Isomorphism

The word isomorphism comes from the Greek word which means 'equal shape'.

An isomorphism is a linear transformation which is both one-to-one and onto.

If two vector spaces V and W have the same basic structures, then there is an isomorphism between them. Dim V = Dim W.

Note: A linear transformation has an inverse if and only if it is an isomorphism.

Example 11.14

State whether the following transformation is an isomorphism or not.

$f: \mathbb{R}^2 \to \mathbb{R}^2: f(x,y) = (x + 2y, 3x + 4y)$

Solution

We have to verify if f is one-to-one and onto.

Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$ then if $f(u) = f(v)$, we have

$(x_1 + 2x_2, 3x_1 + 4x_2) = (y_1 + 2y_2, 3y_1 + 4y_2)$ and the system

$\begin{cases} x_1 + 2x_2 = y_1 + 2y_2 \\ 3x_1 + 4x_2 = 3y_1 + 4y_1 \end{cases}$ and solving, we have $\begin{cases} 2x_1 + 4x_2 = 2y_1 + 4y_2 \\ -3x_1 - 4x_2 = -3y_1 - 4y_2 \end{cases}$ $\quad -x_1 = -y_1$

$x_1 = y_1$ and $x_2 = y_2$. Thus, u = v, and f is one-to-one.

Let (a,b) be an arbitrary element in the range space, then we have to verify if the following system has solution for all values of a and b.

$\begin{cases} x + 2y = a \\ 3x + 4y = b \end{cases}$ $\quad \begin{cases} 2x + 4y = 2a \\ -3x - 4y = -b \end{cases}$ $\quad \begin{cases} -x = 2a - b \\ 2y = a - x \end{cases}$ $\quad \begin{cases} x = b - 2a \\ y = \dfrac{a - x}{2} \end{cases}$

Thus, the transformation is onto.

Hence, the transformation is isomorphism in 2D.

Example 11.15

State whether the following transformation is an isomorphism or not.

$f: \mathbb{R}^2 \to \mathbb{R}^2 : f(x,y) = (x + 2y, 2x + 4y)$

Solution

We have to verify if f is one-to-one and onto.

Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$, then if $f(u) = f(v)$, we have

$(x_1 + 2x_2, 2x_1 + 4x_2) = (y_1 + 2y_2, 2y_1 + 4y_2)$ and the system

$\begin{cases} x_1 + 2x_2 = y_1 + 2y_2 \\ 2x_1 + 4x_2 = 2y_1 + 4y_2 \end{cases}$ and solving, we have $\begin{cases} 2x_1 + 4x_2 = 2y_1 + 4y_1 \\ -2x_1 - 4x_2 = -2y_1 - 4y_2 \end{cases}$

$0x_1 + 0x_2 = 0y_1 + 0y_2$

Some different vectors of the domain have the same image.
Thus f is not one-to-one.
Hence, f is not isomorphism
There is no need to verify if the transformation is onto.

Note: A transformation fails to be isomorphism if it fails to be either one-to-one or onto.

Remark: For the linear application f: V → W when

V = W, it means V → V, then f is called **endomorphism**.

f is one-to-one linear transformation (bijection) then f is called **isomorphism**.

V = W (endomorphism) is one-to one transformation, then f is called **automorphism**.

V = W and f is an injection, then f is called **monomorphism**.

- V = W and f is a surjection, then f is called **epimorphism**.

Note: Injection (one-to-one) : If $f(\vec{a}) = f(\vec{b})$, then $\vec{a} = \vec{b}$.
Surjection (onto) : V → W: For every $f(\vec{a}) \in W$ there exists $\vec{a} \in V$
Bijection verifies both of injection and surjection.

Example 11.16

$f: \mathbb{R}^2 \to \mathbb{R}^2: f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix}$; is f an injection?

Solution

We verify if $f(u) = f(v)$, then $u = v$ where $u\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *and* $v\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$f(u) = f(v) \Rightarrow f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + y_2 \\ y_1 + 3y_2 \end{pmatrix}$$

$$\begin{cases} 2x_1 + x_2 = 2y_1 + y_2 \\ x_1 + 3x_2 = y_1 + 3y_2 \end{cases} \begin{vmatrix} 1 \\ -2 \end{vmatrix}$$

$$\begin{cases} 2x_1 + x_2 = 2y_1 + y_2 \\ -5x_2 = -5y_2 \end{cases} \Rightarrow \begin{cases} 2x_1 + x_2 = 2y_1 + y_2 \\ x_2 = y_2 \end{cases}$$

$$2x_1 = 2y_1 \Rightarrow x_1 = y_1$$

Then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow u = v$

Thus, f is injection

Inverse of linear transformation

Let $f: V \to W$ be a linear transformation such that dim V= dim W i.e. (V = W) $f^{-1}:W \to V$ is also a linear transformation; if $f(f^{-1}(\vec{v})) = \vec{v}$ and $f^{-1}(f(\vec{v})) = \vec{v}$, then we say that f^{-1} is the inverse of f and we say that f is invertible.

Example 11.17

Determine whether $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x,y) = (x + 2y, x - 2y)$ is invertible or not.

Solution

We can simply check whether the standard matrix of f is invertible or not.

In $\mathbb{R}^2$ the standard basis are $\vec{e_1} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{e_2} = \begin{pmatrix}1\\0\end{pmatrix}$ and we have

$f(\vec{e_1}) = f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ and $f(\vec{e_2}) = f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$

The standard matrix of f is $\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix}$

We finally find that $\det A = \begin{vmatrix}1 & 2\\1 & -2\end{vmatrix} = -2 - 2 = -4 \neq 0$.

Thus, M is invertible and hence f is also invertible.

Example 11.18

Let f: $\mathbb{R}^2 \to \mathbb{R}^2$: f(x,y) = (x –2y, x)

Find (a) f^{-1}

(b) $f \circ f^{-1}$

Solution

(a) We let $f(x, y) = (x - 2y, x) = (x', y')$, and we express x and y with in function of x' and y'.

We have: $\begin{cases} x' = x - 2y \\ y' = x \end{cases} \Leftrightarrow \begin{cases} x = y' \\ y = \dfrac{y' - x'}{2} \end{cases}$.

Then a linear application $f^{-1}$is $f^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \to f^{-1}(x, y) = (y, \frac{y - x}{2})$

(b) $f \circ f^{-1}(x,y) = f(f^{-1}(x,y)) = f((y, \frac{y-x}{2}) = (y - 2(\frac{y-x}{2}),y)$

$= (\frac{2y - 2y + 2x}{2},y) = (x, y).$

Application activity 11.3

Verify if the following transformations are isomorphism or not.

1) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x,y) = (3x - y, 2x + y)$
2) $f: \mathbb{R}^2 \to \mathbb{R}^2: f(x,y) = (2x - 4y, -3y + 6y)$

Summary

1. A linear transformation T from a vector space V to a vector space W is a function $T: V \to W$ that satisfies the following two conditions:
   a. (1) : $\forall \vec{u}, \vec{v} \in V$: $\mathbf{T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})}$
   b. (2) : $\forall \vec{u}, \vec{v} \in V$, and scalar α : $\mathbf{T(\alpha \vec{u}) = \alpha T(\vec{u})}$
2. If $T: V \to W$ is a linear transformation then
   a. $T(\vec{0}) = \vec{0}$
   b. $T(-\vec{u}) = -T(\vec{u})$
   c. $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$
   d. $T(\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \ldots + \alpha_n\vec{u}_n) = \alpha_1 T(\vec{u}_1) + \alpha_2 T(\vec{u}_2) + \ldots \alpha_n T(\vec{u}_n)$.
3. If a point (x, y) in the plane is rotated counterclockwise about the origin through an angle θ to obtain a new point (x', y') , then:
   a. $x' = x \cos\theta - y \sin\theta$
   b. $y' = x \sin\theta + y \cos\theta$.
4. If $f: E \to F$ is a linear application (homomorphism), the kernel of f denoted by ker f is the set of all vectors of E which have zero vectors of F as their image.
5. If $f: E \to F$ is a linear application of a vector space E into the vector space F, the range of f denoted by im f is the set of all vectors images in f .
6. The composition of two linear applications is also a linear application.
7. Bijection is the combination of injection and surjection.
8. If $f: V \to W$ is an invertible linear transformation then its inverse is $f^{-1}: W \to V$ and thus $f(f^{-1}(\vec{v})) = \vec{v}$ and $f^{-1}(f(\vec{v})) = \vec{v}$.

Topic area: Linear algebra

Sub-topic area: Linear transformation in 2D

Unit 12

Matrices and determinants of order 2

Key unit competence

Use matrices and determinants of order 2 to solve systems of linear equations and to define transformations of 2D.

12.0 Introductory activity

A Farmer Kalisa bought 5 Cocks and 4 Rabbits and he paid 35,000 FRW in Ruhango Market, and the following day, he bought 3 Cocks and 6 Rabbits and he paid 30,000 FRW in the same Market.

a) Arrange what Kalisa bought according to their types in a simple table as follows

Cocks	Rabits	Price

b) Discuss and explain in your own words how you can determine the cost of 1 cock and the cost of 1 Rabbit.

12.1 Introduction

Activity 12.1

In pairs, carry out research to find out the meaning of the term matrix. What is the plural of matrix? Where and when can we make use of a matrix?

A matrix is an ordered set of numbers listed in rectangular form.

Order of a matrix

The number of rows and columns that a matrix has is called its **order** or its **dimension**. By convention, rows are listed first; and columns, second. Thus, we would say that the order (or dimension) of the matrix below is 3 x 4, meaning that it has 3 rows and 4 columns.

$$\begin{bmatrix} 2 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

We denote the element on the second row and fourth column with $a_{2,4}$.

Let us take an example:

$$B = \begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

Here are some sample entries:

$b_{1,1} = 6$ *(the entry at row 1, column 1 is 6)*

$b_{1,3} = 24$ *(the entry at row 1, column 3 is 24)*

$b_{2,3} = 8$ *(the entry at row 2, column 3 is 8)*

Square matrix

If a matrix A has n rows and n columns then we say it is a square matrix. In a square matrix the elements $a_{i,i}$, with i = 1, 2, 3,... , are called **diagonal elements**.

Note: There is no difference between a 1 x 1 matrix and an ordinary number.

Diagonal matrix

A diagonal matrix is a square matrix with all non-diagonal elements 0. The diagonal matrix is completely defined by the diagonal elements.

Example 12.1

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The matrix is denoted by diag(7 , 5 , 6)

Row matrix

A matrix with one row is called a row matrix.

$$\begin{bmatrix} 2 & 5 & -1 & 5 \end{bmatrix}$$

Column matrix

A matrix with one column is called a column matrix.

$$\begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}$$

We talk about one **matrix**, or several **matrices**.

There are many things we can do with them.

12.2 Matrix of a linear transformation

Let $f:\mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x,y) \rightarrow f(x,y) = A(x,y)$ be a linear transformation in two dimension. A is called a matrix of a linear transformation.

Example 12.2

Let $f:\mathbb{R}^2 \rightarrow \mathbb{R}^2: (x, y) \rightarrow f(x, y) = (x + y, x + 2y)$ be a linear transformation. Find the matrix of transformation.

Solution

$A = [f(\overrightarrow{e_1}), f(\overrightarrow{e_2})]$

$f(\overrightarrow{e_1}) = f(1, 0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$f(\overrightarrow{e_2}) = f(0, 1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is a matrix of a linear transformation.

12.3 Matrix of geometric transformation

Symmetric about x-axis and y-axis

For the symmetry about x-axis

$T(\overrightarrow{e_1}) = \overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \overrightarrow{e_1} + 0\overrightarrow{e_2}$ and $T(\overrightarrow{e_2}) = -\overrightarrow{e_2} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0\overrightarrow{e_1} + 0\overrightarrow{e_2}$

The transformation matrix is $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Thus, by the symmetry in x-axis, the matrix $M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ transforms the point $\vec{u} =(x, y)$ into $\vec{u}' = (x', y')$ as: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

$\vec{u} = T(\vec{u}) = T(x\overrightarrow{e_1} + y\overrightarrow{e_2}) = xT(\overrightarrow{e_1}) + y(\overrightarrow{e_2}) = x\overrightarrow{e_1} - y\overrightarrow{e_2} = \begin{pmatrix} x \\ -y \end{pmatrix}$

For the symmetry about y-axis

$T(\overrightarrow{e_1}) = -\overrightarrow{e_1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\overrightarrow{e_1} + 0\overrightarrow{e_2}$ and $T(\overrightarrow{e_2}) = \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\overrightarrow{e_1} + \overrightarrow{e_2}$

The transformation matrix is $M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus, By the symmetry in y-axis, the matrix $M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ transforms the point $\vec{u} = (x,y)$ into $\vec{u}' = (x', y')$ as: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$

$\vec{u}' = T(\vec{u}) = xT(\vec{e_1}) + yT(\vec{e_2}) = x\vec{e_1} + y\vec{e_2} = \begin{pmatrix} -x \\ y \end{pmatrix}$

Rotation

In matrix notation, this can be written $\begin{pmatrix} x' \\ y' \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

A $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a matrix of a linear transformation.

Example 12.3

Find the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each of the vectors $\vec{e_1}$ and $\vec{e_2}$ counterclockwise 90°. Then explain why T rotates all vectors in $\mathbb{R}^2$ counterclockwise 90°.

Solution

The T we are looking for must satisfy both

$T(\vec{e_1}) = T\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(\vec{e_2}) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The standard matrix for T is thus

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

and we know that $T(\vec{x}) = A\vec{x}$ for all $x \in \mathbb{R}^2$. Hence, for any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we have $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

By using some basic trigonometry, we can see that $T(\vec{x})$ is $\vec{x}$ rotated counterclockwise 90°.

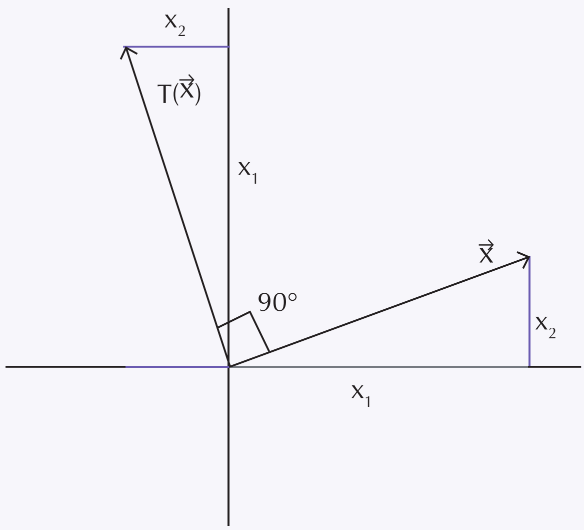


Fig 12.1

Activity 12.2

In pairs, work out the following:

Find the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that perpendicularly projects both of the vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ onto the line $x_2 = x_1$. Then explain why T perpendicularly projects all vectors in $\mathbb{R}^2$ onto the line $x_2 = x_1$.

Make use of basic trigonometry, using diagrams, to verify your findings.

12.4 Operations on matrices

A matrix is a rectangular array of elements that are stored in rows and columns of a given order. The matrix of order 2 (2 x 2) is in the form $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where a_{11}, a_{21}, a_{12} and a_{22} are called "the elements of the matrix".

Equality of matrices

Activity 12.3

Carry out research to find out the meaning of equal matrices. Discuss in class using examples.

For two matrices to be equal, they must be of the same size and have all the same entries in the same places. For instance, suppose you have the following two matrices:

$A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 0 \end{bmatrix}$

These matrices cannot be the same, since they are not of the same size.

If *A* and *B* are the following two matrices:

$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

A is a 3 x 2 matrix and *B* is a 2 x 3 matrix, and, for matrices, 3 x 2 does not equal 2 x x 3! It does not matter if *A* and *B* have the same number of entries or even the same numbers as entries. Unless *A* and *B* are the same size and the same shape and have the same values in exactly the same places, they are not equal.

Example 12.4

Given that the following matrices are equal, find the values of *x* and *y*.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} x & 2 \\ 3 & y \end{bmatrix}$

Solution

For A and B to be equal, they must have the same size and shape (which they do; they're each 2 x 2 matrices) and they must have the same values in the same positions. Then $a_{1,1}$ must equal $b_{1,1}$, $a_{1,2}$ must equal $b_{1,2}$, and so forth. The entries $a_{1,2}$ and $a_{2,1}$ are clearly equal, respectively, to entries $b_{1,2}$ and $b_{2,1}$ "by inspection". But $a_{1,1} = 1$ is not obviously equal to $b_{1,1} = x$. For A to equal B, we must have $a_{1,1} = b_{1,1}$, so it must be that $1 = x$. Similarly, we must have $a_{2,2} = b_{2,2}$, so then 4 must equal y. Then the solution is:

$x = 1, \qquad y = 4$

Application activity 12.1

Given that the following matrices are equal, find the values of x, y, and z.

$$A = \begin{pmatrix} 4 & 0 \\ 6 & -2 \\ 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} x & 0 \\ 6 & y+4 \\ \frac{z}{3} & 1 \end{pmatrix}$$

Addition and subtraction of matrices

Addition and subtraction of matrices is only possible for matrices of the same order.

Addition

To add two matrices: add the numbers in the matching positions:

3 + 4 = 7

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

These are the calculations:

$3 + 4 = 7 \qquad 8 + 0 = 8$

$4 + 1 = 5 \qquad 6 - 9 = -3$

The two matrices must be of the same size, i.e. the rows must match in size, and the columns must match in size.

General given that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$; then

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Example 12.5

Given that matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and matrix $B = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$, find

(a) A + B

(b) A – B.

Solution

(a) $A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1+4 & 2-2 \\ 3+1 & 4+3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 4 & 7 \end{pmatrix}$

(b) $A - B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1-4 & 2+2 \\ 3-1 & 4-3 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 2 & 1 \end{pmatrix}$

Note: A square matrix has the same numbers of rows and columns for example a 2 x 2 is a square matrices of order 2.

Subtraction

Negative

The negative of a matrix is also simple:

– (2) = –2

$$-\begin{bmatrix} 2 & -4 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -7 & -10 \end{bmatrix}$$

These are the calculations:

– (2) = –2   – (–4) = +4

– (7) = –7   – (10) = –10

Subtracting

To subtract two matrices: subtract the numbers in the matching positions:

3 – 4 = –1

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

These are the calculations:

3 – 4 = –1   8 – 0 = 8

4 – 1 = 3   6 – (–9) = 15

Generally given that $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$; then

$$A - B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$$

Note: Subtracting is actually defined as the **addition** of a negative matrix: A + (–B)

Application activity 12.2

Work out:

What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} - \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

Multiplication of a matrix by a scalar

To multiply a matrix by a single number is easy:

2 x 4 = 8

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

2 x 4 = 8   2 x 0 = 0

2 x 1 = 2   2 x –9 = –18

We call the number (2 in this case) a **scalar**, so this is called scalar multiplication.

If matrix A is multiplied by a scalar k then all the elements of A are multiplied by the scalar that is if ,

$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$kA = k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$.

Example 12.6

Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and B $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$; determine 2A + B

Solution

$$2A + B = 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 2+5 & 4+6 \\ 6+7 & 8+8 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 13 & 16 \end{pmatrix}$$

Identity matrix

An identity matrix of order 2 has elements in the leading diagonal as 1 and the elements in the secondary diagonal are zero, it is written as $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Multiplication of matrices

Two matrices can be multiplied if and only if the number of columns of the first matrix is equal to the number of rows of the second matrix. Such matrices are said to be compatible to multiplication.

To multiply a matrix **by another matrix** we need to use the **dot product** of rows and columns.

Let us see using an example:

To work out the answer for the **1st row** and **1st column**:

"Dot product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \\ & \end{bmatrix}$$

The "dot product" is where we **multiply matching members**, then sum up:

(1, 2, 3)   (7, 9, 11) = 1x7 + 2x9 + 3x11 = 58

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

Let us see another example. For the 1st row and **2nd column**:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

(1, 2, 3)   (8, 10, 12) = 1 x 8 + 2 x 10 + 3 x 12 = 64

We can do the same thing for the **2nd row** and **1st column**:

(4, 5, 6)   (7, 9, 11) = 4x7 + 5x9 + 6x11 = 139

And for the **2nd row** and **2nd column**:

(4, 5, 6)   (8, 10, 12) = 4 x 8 + 5 x 10 + 6 x 12 = 154

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Example 12.7

Given A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$; determine

(a) AB

(b) BA

Solution

(a) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} (1)(5)+(2)(7) & (1)(6)+(2)(8) \\ (3)(5)+(4)(7) & (3)(6)+(4)(8) \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

(b) $BA = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} (5)(1)+(6)(3) & (5)(2)+(6)(4) \\ (7)(1)+(8)(3) & (7)(2)+(8)(4) \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$

Transpose of a matrix

The transpose of the matrix A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is denoted $A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$.

The rows become the columns and the columns become the rows.

Application activity 12.3

If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, what is AB?

If $A = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}$, what is AB?

If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$, what is QP?

If $A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & -1 & 6 \\ -4 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & -5 & 2 \\ 1 & 4 & -2 \end{bmatrix}$, what is BA?

12.5 Determinant of a matrix of order 2

Activity 12.4

In pairs, find out the meaning of the term determinant as used in the case of matrices. Why do we need determinants? What is the symbol for determinant?

The **determinant** of a matrix is a special number that can be calculated from a square matrix. The symbol for determinant is two vertical lines either side. For example,

|A| means the determinant of the matrix **A**.

The determinant of the 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is denoted and defined by $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant is a scalar (number).

Example 12.8

Find the determinant of the matrix $A = \begin{pmatrix} 3 & 5 \\ -2 & -1 \end{pmatrix}$

Solution

$\det A = \begin{vmatrix} 3 & 5 \\ -2 & -1 \end{vmatrix} = 3(-1) - 5(-2) = -3 + 10 = 7$

Note A matrix A for which det A = 0 is called a **singular matrix**.

Example 12.9

For what values of a is the matrix $A = \begin{pmatrix} 3-a & 8 \\ 1 & -4-a \end{pmatrix}$ singular?

Solution

$\det A = 0 \Leftrightarrow$

$\det A = (3-a)(-4-a) - 8 = -12 - 3a + 4a + a^2 - 8 = a^2 + a - 20 = (a+5)(a-4) = 0$

Thus A is singular when a = –5 or a = 4.

Inverse of matrices

If A is a **square** matrix then a matrix B such that $AB = BA = I_2$ is called an **inverse** of the matrix A.

Note that the inverse of A is denoted by A^{-1}. A matrix A which has an inverse is called **a non-singular** or **invertible** matrix.

A matrix A is said to be invertible if $\det A \neq 0$.

If A and B are invertible then we can write $(AB)^{-1} = B^{-1}A^{-1}$.

Consider the 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its inverse, if it exists, then:

$A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided $\det A \neq 0$.

Example 12.10

Find the inverse of the matrix $A = \begin{pmatrix} 8 & 3 \\ 6 & 2 \end{pmatrix}$

Solution

$\det A = 16 - 18 = -2$ $\quad A^{-1} = \frac{1}{-2}\begin{pmatrix} 2 & -3 \\ -6 & 8 \end{pmatrix} = \begin{pmatrix} -1 & \frac{3}{2} \\ 3 & -4 \end{pmatrix}$

Example 12.11

Show that $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ are inverse matrices to each other.

Solution

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Thus we can say that A and B are inverses of each other.

Application activity 12.4

1. Given $A = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}$, find 2A, –A and $\frac{1}{2}A$.
2. For matrices $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$, find AB.
3. Suppose $B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$; compute B^0, B^2 and B^3.
4. Find det A, where $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$.
5. Find det D, where $D = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$.
6. For matrices $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$, show that $AB \neq BA$.
7. Show that the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ does not have an inverse but matrix $B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ does. Find the matrix B^{-1}.

Application of determinants

Cramer's method

To solve simultaneous equation, we proceed as follows:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix};\ D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix};\ D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$

The solution is (x, y).

Example 12.12

Solve the following system of equations $\begin{cases} x + y = 8 \\ x - y = 2 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$D_x = \begin{vmatrix} 8 & 1 \\ 2 & -1 \end{vmatrix} = -8 - 2 = -10$$

$$D_y = \begin{vmatrix} 1 & 8 \\ 1 & 2 \end{vmatrix} = 2 - 8 = -6$$

$$x = \frac{D_x}{D} = \frac{-10}{-2} = 5$$

$$y = \frac{D_y}{D} = \frac{-6}{-2}, = 3$$

The solution to the system is $S = \{(x, y)\} = \{(5, 3)\}$

Example 12.13

A painter works x hours at the rate of 100 FRW per hour and y hours overtime at a rate of 120 FRW per hour. The painter altogether works for 60 hours, and his total earnings are 7,000 FRW. Determine x and y.

Solution

Here one equation should show the total earnings while the other should show the total number of hours worked.

Total earnings for working x hours and y hours:

$100x + 120y = 7,000$...............(i)

Total number of hours:

$x + y = 60$.........................(ii)

Multiply (ii) by 100 to make coefficients of x equal.

$100(x + y = 60)$, which gives

$100x + 100y = 6,000$.............(iii)

Now, solve (i) and (iii)

$$\begin{cases} 100x + 120y = 7,000 \\ 100x + 100y = 6,000 \end{cases}$$

$$D = \begin{vmatrix} 100 & 120 \\ 100 & 100 \end{vmatrix} = 10,000 - 12,000 = -2,000$$

$$D_x = \begin{vmatrix} 7,000 & 120 \\ 6,000 & 100 \end{vmatrix} = 700,000 - 720,000 = -20,000$$

$$D_y = \begin{vmatrix} 100 & 7,000 \\ 100 & 6,000 \end{vmatrix} = 600,000 - 700,000 = -100,000$$

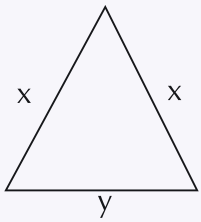
$$x = \frac{D_x}{D} = \frac{-20,000}{-2,000} = 10$$

$$y = \frac{D_y}{D} = \frac{-100,000}{-2,000} = 50$$

Therefore, the painter works for 10 hours earning 100 FRW per hour and 50 hours at a rate of 120 FRW per hour.

Application activity 12.5

1. Nkusi Drug Store sold 2 tins of Panadol and 4 tins of Fansidar tablets. The total sales of the two drugs amounted to 108,000 FRW. The following day, 3 tins of both drugs were sold at 126,000 FRW. What was the unit-selling price of Fansidar and Panadol tablets?
2. The length of the sides of an isosceles triangle are x cm and y cm respectively. The longer side (x) is twice as long as its base (y). If the perimeter of the triangle is 30 cm, determine x and y.



3. BCK supermarket is selling items at giveaway prices during its sales promotion. Bacon costs x FRW per packet and sausages cost y FRW per piece. The total cost of one packet of bacon and 4 pieces of sausage is 9,100 FRW. Two packets of bacon and one piece of sausage cost 10,500 FRW. Find the cost of one packet of bacon and one sausage (piece).
4. A charter plane is providing air service for tourists to a national park. The tourists can either book first class or third class. First class fare is x FRW per tourist, while third class is y FRW per tourist. If 20 tourists travel in first class and 40 tourists in third class, the total transport cost is 1,080,000 FRW. When 30 tourists travel first class and 30 tourists in third class, the transport cost increase to 1,185,000 FRW. Determine the air fare for first class and third class.

Summary

1. If f: $\mathbb{R}^2 \to \mathbb{R}^2$: $(x,y) \to f(x,y) = A(x,y)$ is a linear transformation in two dimension, then A is called a **matrix of a linear transformation**.
2. A matrix is a **rectangular array** of elements that are stored in rows and column of a given order. The matrix of order 2 (2x2) is in the form

   $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where a_{11}, a_{21}, a_{12}, a_{22} are called "the elements of a matrix".
3. Addition and subtraction of matrices is only possible for matrices of the same order.
4. A **square matrix** has the same numbers of rows and columns.
5. If matrix A is multiplied by a scalar k then all the elements of A are multiplied by the scalar.
6. Two matrices can be multiplied if and only if the number of columns of the first matrix is equal to the number of rows of the second matrix.
7. The **transpose** of the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is denoted $A^T = \begin{pmatrix} a_{21} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$
8. The **determinant** of the 2 x 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is denoted and defined by $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
9. A matrix A for which det A = 0 is called a **singular matrix**.

Topic area: Geometry

Sub-topic area: Plane geometry

Unit 13

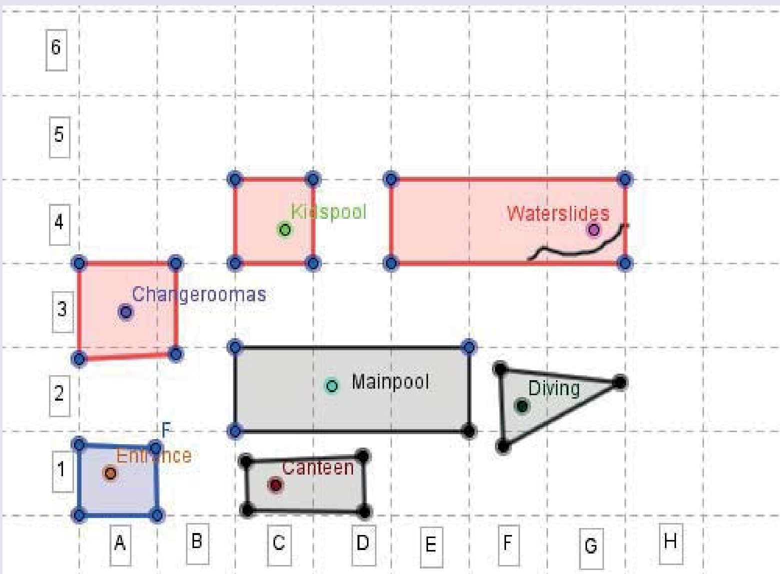
Points, straight lines and circles in 2D

Key unit competence

Determine algebraic representations of lines, straight lines and circles in 2D.

13.0 Introductory activity

Look at this map of a swimming centre.



a) What feature is located at the map reference A3?

b) How many grid squares does the main pool cover?

c) What is the map reference for the:

i) Diving area?

ii) Canteen?

d) In which direction are the:

i) Water slides from the kids' pool?

ii) Change rooms from Canteen?

iii) Main pool from diving?

13.1 Points in 2D

Activity 13.1

What is a point?

In Junior Secondary, we studied points on a Cartesian plane. How do we define the coordinate of a point in 2D? Discuss in groups and present your findings to the rest of the class

A point is an exact position or location on a plane surface. It is important to understand that a point is not a thing, but a place. We indicate the position of a point by placing a dot with a pencil. In coordinate geometry, points are located on the plane using their coordinates - two numbers that show where the point is positioned with respect to two number line "axes" at right angles to each other.

Cartesian coordinate of a point

In a 2-dimensional plane, representing position vectors of point I (1,0) and J (0,1) respectively, each point in such a plane can be represented in terms of unit vectors. The position vector of R (x, y) is (x,y) is $r = x\vec{i} + y\vec{j}$

where $\vec{i}(1, 0)$ and $\vec{j}\ (0,1)$

$|\vec{i}| = \sqrt{1^2 + 0^2} = 1$ and $|\vec{j}| = \sqrt{0^2 + 1^2} = 1$

Let P(a, b) and Q(c, d) be any points in the following Cartesian plane, with position vector $\overrightarrow{OP} = (a, b)$ and $\overrightarrow{OQ} = (c, d)$, respectively, as illustrated below:

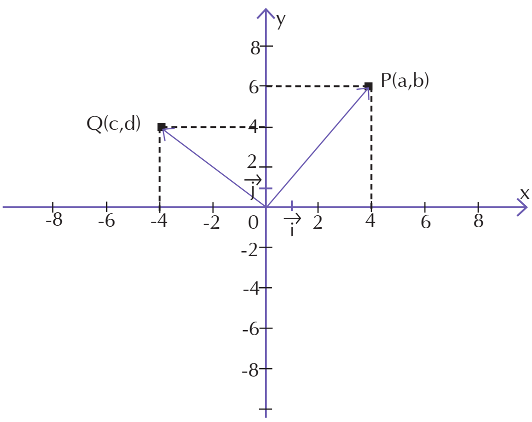


Fig 13.1

We see that the position vectors $\overrightarrow{OP} = (4, 6) = 4\vec{i} + 6\vec{j}$ and $\overrightarrow{OQ} = (-4, 4) = -4\vec{i} + 4\vec{j}$, respectively.

Distance between two points

If A (x_1, y_1) and B(x_2, y_2) are two points, then the length of the line joining points A and B, i.e the distance, is:

$$d(\overrightarrow{AB}) = |\overrightarrow{AB}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example 13.1

The length of a line segment joining A (3, y) and B (7, 5) is 5 units. Find the possible values of y.

Solution

$d(AB) = \sqrt{(7-3)^2 + (5-y)^2} = 5$

$\sqrt{16 + 25 - 10y + y^2} = 5$

$25 = 41 - 10y + y^2$

$y^2 - 10y + 16 = 0$

$\Delta = b^2 - 4ac = 100 - 4(16)(1) = 100 - 64 = 36$

$y = \frac{-b + \sqrt{\Delta}}{2a} = \frac{10 + 6}{2(1)} = \frac{16}{2} = 8$ or $y = \frac{-b - \sqrt{\Delta}}{2a} = -\frac{10 - 6}{2(1)} = \frac{4}{2} = 2$

The possible values of y are 8 and 2.

The mid-point of a line segment

Let A (x_1, y_1), B (x_2, y_2) be two points. To find the coordinates of the mid-point (M) of a vector $\overrightarrow{AB}$:

$M = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

Therefore, the coordinates of point M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Therefore, the formula for the coordinates of a mid-point of a line is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 13.2

Given the points A(2, –1) and B(6, 5), calculate:

(a) the distance between points A and B

(b) the mid-point of the line joining A and B.

Solution

(a) The distance between points A and B is:

$|\overrightarrow{AB}| = \sqrt{(6-2)^2 + (5-(-1))^2} = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52}$

(b) The mid-point of the line is:

$M = \left(\frac{2+6}{2}, \frac{-1+5}{2}\right) = \left(\frac{8}{2}, \frac{4}{2}\right) = (4,2)$

Application activity 13.1

1. Find the coordinates of the mid-point of the straight lines joining each of the following pair of points:

   a) (7, 2) and (5, 8) b) (4, –2) and (2, 4)

   c) (1, 7) and (6, 3) d) (0, 6) and (2, 0)

   e) (–8, –6) and (–2, –4) f) (–5, 7) and (13, –3)

   g) (–3, 8) and (–15, 0)

2. Find the position vector of the mid-point of the line joining two points whose position vectors are as follows:

   a) $\begin{pmatrix}5\\5\end{pmatrix}$ and $\begin{pmatrix}3\\7\end{pmatrix}$ b) $\begin{pmatrix}-2\\-1\end{pmatrix}$ and $\begin{pmatrix}6\\7\end{pmatrix}$

   c) $\begin{pmatrix}-2\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\-1\end{pmatrix}$

   d) $2\vec{i} + 3\vec{j}$ and $4\vec{i} - 3\vec{j}$ e) $2\vec{i} - \vec{j}$ and $10\vec{i} + 5\vec{j}$

   f) $-3\vec{i} + 2\vec{j}$ and $-5\vec{i} - 6\vec{j}$

3. M is a mid-point of a straight line joining point A(10, 5) to point B. If M has coordinates (6, 4), find the coordinates of B.

4. N(7, –3) is a mid-point of a straight line joining point C(p, –5) to point D(9, q). Find q and p.

5. Find the coordinate of the point E, if the point F(–10, 4) is the mid-point of the straight line joining E to G(–11, 16).

6. Given the triangle with vertices at A(2, 4), B(4, –2) and C(8, 12) and that L is the mid-point of $\overrightarrow{AB}$ and M is the mid-point of $\overrightarrow{BC}$, find:

   a) the coordinate of point L b) the coordinate of point M

   c) the distance LM.

7. If a triangle has vertices at A(9, 9), B(3, 2) and C(9, 4), find

   a) the coordinate of M, the mid-point of $\overrightarrow{BC}$

   b) the length of the median from A to M.

8. Find the length of the medians of the triangle that has vertices at A(0, 1), B(2, 7) and C(4, –1).
9. If a parallelogram PQRS has vertices P(–1, 5), Q(8, 10), R(7, 5) and S, Calculate the coordinates of S.
10. A(1, 1), B(2, 7) and C(13, 7) and D are the vertices of the parallelogram ABCD. Find:
    a) the coordinates of D
    b) the coordinates of the mid-point of the diagonal DB
    c) the coordinates of the mid-point of the diagonal AC.

13.2 Lines in 2D

The equations of straight lines

A particular line is uniquely located in a plane if

it has a known direction and passes through a known fixed point, or

it passes though two known points.

Cartesian equation of a straight line

Equation of a line through points A(x_1, y_1) and B(x_2, y_2)

Consider a line which passes through two points A(x_1, y_1) and B(x_2, y_2) and suppose P(x, y) is any other point on the line.The gradient of AP is equal to the gradient of AB as ABP is a straight line:

so $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

Note the equation of a line can be written as

$(y - y_1) = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$; here $\frac{y_2-y_1}{x_2-x_1}$ is a gradient of the straight line

Example 13.3

Find the equation of the straight line that passes through (3, –1) and (7, 2).

Solution

Let P(x, y) be any point on the line and use $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$.

$\frac{y-(-1)}{x-3} = \frac{2-(-1)}{7-3}$ or $4(y + 1) = 3(x - 3)$, which gives $3x - 4y - 13 = 0$.

Equation of a line given its gradient and a point

Given a point A (x_1, y_1) and a gradient m, then the equation of a line is given by $(y - y_1) = m(x - x_1)$.

Example 13.4

Find the equation of the line whose gradient is 3 and passes through the point (–1, 2).

Solution

From the equation $(y - y_1) = m(x - x_1)$

$(y - 2) = 3\ (x - (-1))$

$y - 2 = 3(x + 1)$

$y - 2 = 3x + 3$

$3x - y + 5 = 0$

Equation of a line given a law

We can find the equation of the locus by considering a point P(x, y) on the locus and using the law to derive an equation in x and y. This will be the equation of the locus.

Example 13.5

Find the equation of the locus of the points which are equidistant from A(3,–2) and B(–4, 1).

Solution

Suppose that P(x, y) is any point on the required locus. Since P is equidistant from A and B,

PA = PB and therefore $|PA|^2 = |PB|^2$

$(x - 3)^2 + (y - (-2))^2 = (x - (-4))^2 + (y - 1)^2$

$(x - 3)^2 + (y + 2)^2 = (x + 4)^2 + (y - 1)^2$

$x^2 - 6x + 9 + y^2 + 4y + 4 = x^2 + 8x + 16 + y^2 - 2y + 1$

$-6x + 4y + 13 = 8x - 2y + 17$

$14x - 6y = -4$

$y = \frac{7}{3}x + \frac{2}{3}$

Parallel and perpendicular lines

If two lines are parallel, they have equal gradients
If two lines are perpendicular, the product of their gradients is –1.

Example 13.6

The triangle with points P(–4, 3), Q(–1, 5), and R(0, –3) is right angled. Find which side is the hypotenuse.

Solution

Gradient PQ = $\frac{5-3}{-1-(-4)} = \frac{2}{3}$

Gradient PR = $\frac{-3-3}{0-(-4)} = -\frac{6}{4} = -\frac{3}{2}$

Gradient QR = $\frac{-3-5}{0-(-1)} = -8$

(gradient PQ) x (gradient PR) = $\frac{2}{3} \times -\frac{3}{2} = -1$

Hence line PQ is perpendicular to line PR. So angle QPR = 90° and line QR is the hypotenuse.

Application activity 13.2

1. Find the equation of the line that passes through the following points:
   a) (1, 2) and (0, 2)
   b) (1, 3) and (–2, 5)
   c) (–3, 0) and (1, 4)
   d) (8, 2) and (3, –5)
   e) (4, –2) and (6, –1).
2. Find the equation of the line with the given point and gradient:
   a) Point (2, 3), gradient 2
   b) Point (1, 5), gradient –1
   c) Point (–3, 0), gradient 3
   d) Point (4, 2), gradient 1
   e) Point (9, –3), gradient –2
3. Find the equation of the straight line that passes through the points A and B, where A is the midpoint of the straight line joining C(1, 3) to D(4, 2) and B is the midpoint of line joining E(–3, –1) to F(–1, 5).
4. Find the equation of the straight line passing through the points that are equidistant from the points A(2, 4) and B(–1, 3).

5. Given two points A(3, 1) and B(4, 8). Find

   a) the mid-point of $\overrightarrow{AB}$ b) the gradient of $\overrightarrow{AB}$

   c) the equation of the line which is a perpendicular bisector of $\overrightarrow{AB}$.

6. A triangle has vertices at A(0, 7), B(9, 4) and C(1, 0). Find

   a) the equation of the perpendicular line from C to $\overrightarrow{AB}$

   b) the equation of the straight line from A to the mid-point of $\overrightarrow{BC}$.

7. Find the equations of the medians of the triangle with vertices at A(1, 0), B(5, 2) and C(1, 6).

Vector, parametric, scalar and Cartesian equations of the line

In 2D, a line can be defined by an equation in slope–intercept form, a vector equation, parametric equations, or a Cartesian equation (scalar equation).

Slope intercept form	$y = ax + b$	a is the slope of the line b is the y-intercept
Vector	$\vec{r} = \vec{r_0} + t\vec{v}$, $t \in \mathbb{R}$ Or $[x,y] = [x_0,y_0] + t[v_1,v_2]$, $t \in \mathbb{R}$	$\vec{r} = [x,y]$ is a position vector to any point on the line $\vec{r_0} = [x_0,y_0]$ is a position vector to a known point on the line j $\vec{v} = [v_1,v_2]$ is a direction vector for the line
Parametric	$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \end{cases}$, $t \in \mathbb{R}$	
Symmetric	$\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2}$ (=t)	provided none of v_1 or v_2 is 0.
Cartesian	$Ax + By + C = 0$	$\vec{n} = [A,B]$ is a normal vector to the line

Example 13.7

Given the vector equation of the line below:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

write

(a) its parametric equation

(b) its Cartesian equation.

Solution

(a) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 2+3t \\ -1-2t \end{pmatrix}$ which gives $x = 2 + 3t$ and $y = -1 - 2t$.

So, the parametric equation is $\begin{cases} x = 2 + 3t \\ y = -1 - 2t \end{cases}$

(b) Making t the subject of each equation gives $t = \dfrac{x-2}{3}$ and $t = \dfrac{y+1}{-2}$

Hence $\dfrac{x-2}{3} = \dfrac{y+1}{-2}$ and so $-2x + 4 = 3y + 3$ $2x + 3y - 1 = 0$.

So, the Cartesian equation is $2x + 3y - 1 = 0$

Example 13.8

Find a vector equation of the line that passes through the point with position vector $2\vec{i} - \vec{j}$ and is parallel to the vector $\vec{i} + \vec{j}$.

Solution

The vector equation of a line is $r = \vec{a} + \lambda\vec{m}$ where a is the position vector of a point on the line and $\vec{m}$ is parallel to the line. For this line, $\vec{a} = 2\vec{i} - \vec{j}$ and m $= \vec{i} + \vec{j}$.

A vector equation of the line is $r = 2\vec{i} - \vec{j} + \lambda(\vec{i} + \vec{j})$.

Example 13.9

Find the vector equation for the line through the points A(3, 4) and B(1, -1).

Solution

To find a vector equation of a line we need a point on the line (we can use either A or B) and a vector parallel to the line.

Since points A and B are on the line, $\overrightarrow{AB}$ is parallel to the line and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

$\overrightarrow{AB} = (\vec{i} - \vec{j}) - (3\vec{i} + 4\vec{j}) = (1 - 3)\vec{i} + (-1 - 4)\vec{j} = -2\vec{i} - 5\vec{j}$

The vector equation of the line is $\vec{r} = 3\vec{i} + 4\vec{j} + \lambda(-2\vec{i} - 5\vec{j})$.

Intersection of lines with equations in Cartesian form

Any point on a line has coordinates which will satisfy the equation of that line. In order to find the point in which two lines intersect we have to find a point with coordinates which satisfy both equations. This is equivalent, from an algebraic point of view, to solving the equations of the lines simultaneously.

Example 13.10

Find the coordinates of the point in which the lines $x - 3y = 1$ and $2x = 5y + 3$ intersect.

Solution

Since the point of intersection satisfies both equations.

$\begin{cases} x - 3y = 1 \\ 2x = 5y + 3 \end{cases}$ $\begin{cases} x - 3y = 1 \\ 2x - 5y = 3 \end{cases}$ $\begin{cases} -2x + 6y = -2 \\ 2x - 5y = 3 \end{cases}$ $\begin{cases} y = 1 \\ x = 3y + 1 \end{cases}$ $\begin{cases} y = 1 \\ x = 4 \end{cases}$

Therefore, the point of intersection is (4, 1).

Application activity 13.3

Find the coordinates of the points where the following pairs of lines intersect:

a) $x + 2y = 2$ and $3x - 2y = 14$

b) $y + 2x = 5$ and $y = 3x - 5$

c) $y = 2x - 1$ and $y = x + 1$

d) $y = x + 2$ and $2y = 3x + 1$.

The intersection of two lines with equations given in vector form

In order to find the point of intersection of two lines whose equations are given in vector form each equation must have a separate parameter. The method as illustrated in the following example:

Example 13.11

Find the point of intersection of the lines $r = 3\vec{i} + \vec{j} + t_1(2\vec{i} - \vec{j})$, and $r = 5\vec{i} - 12\vec{j} + t_2(\vec{i} + \vec{j})$.

(These lines must meet since their direction vector, $2\vec{i} - \vec{j}$ and $\vec{i} + \vec{j}$ are not parallel.)

Solution

By equating the components in the first line we have $x = 3 + 2t_1$; $y = 1 - t_1$ and in the second line we have $x = 5 + t_2$; $y = -12 + t_2$.

For intersection $3 + 2t_1 = 5 + t_2$ and $1 - t_1 = -12 + t_2$.

Solving these equations simultaneously we find that $t_1 = 5$ and $t_2 = 8$ hence these lines meet at the point with position vector $13\vec{i} - 4\vec{j}$.

13.3 Points and lines

Distance of a point from a line

The perpendicular distance from a point $P(x_1, y_1)$ to the line $ax + by + c = 0$ is given by

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

Example 13.12

Find the perpendicular distance from the line $2x + y - 1 = 0$ to the point $(3, 2)$.

Solution

The distance is

$$\frac{2(3) + 1(2) - 1}{\sqrt{(2)^2 + (1)^2}} = \frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{5}$$

Note: Two lines L_1 and L_2 with vector equations $L_1 \equiv r = \vec{r_1} + t_1\,\vec{v_1}$ and $L_2 \equiv r = \vec{r_2} + t_2\,\vec{v_2}$ respectively are parallel if they have the same direction vectors i.e $\vec{v_1} = \vec{v_2}$.

Example 13.13

Find the vector equation of the line that passes through the point $(1, 2)$ and parallel to the line of equation $\begin{cases} x = 3 - 2t \\ y = 4 + 5t \end{cases}, t \in \mathbb{R}$

Solution

The vector form of the given line is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

So the vector equation of the required line is $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + m\begin{pmatrix} -2 \\ 5 \end{pmatrix}, m \in \mathbb{R}$

Angle between two straight lines

Consider the two lines with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively and suppose that they make an angle of θ_1 and θ_2 respectively with the positive x-axis. Let θ be an acute angle between the two lines,

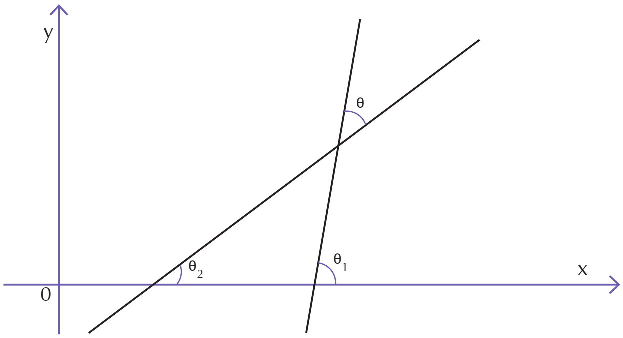


Fig 13.2

Then

$\theta = \theta_1 - \theta_2$ i.e. $\tan\theta = \tan(\theta_1 - \theta_2) = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$

But $\tan\theta_1$ and $\tan\theta_2$ are the gradients of these lines and hence $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Hence, $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$, note that the angle θ between the lines depends only on the gradients of the lines; the constants c_1 and c_2 do not affect the angle θ.

Example 13.14

Find the tangent of the acute angle between the pair of lines whose equations are $3y = x - 7$ and $2y = 3 - 4x$.

Solution

The pair of lines are:

$y = \frac{1}{3}x - \frac{7}{3}$, gradient is $\frac{1}{3} = m_1$; $y = \frac{3}{2} - 2x$, gradient is $-2 = m_2$

If the angle between the lines is θ, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta = \frac{\frac{1}{3} - (-2)}{1 + \left(\frac{1}{3}\right)(-2)} = \frac{\frac{1+6}{3}}{\frac{3-2}{3}} = \frac{\frac{7}{3}}{\frac{1}{3}} = 7\,.$$

Thus, $\tan\theta = 7$, the tangent of the acute angle between the two lines is 7.

Application activity 13.4

1. Find the tangent of the acute angle between the following pairs of lines:
   a) $y = 2x - 3$ and $y = x + 4$
   b) $y = 2x + 4$ and $2y = x - 10$
   c) $4y = 3x - 4$ and $y + 2x = 3$
2. Calculate the angles of the triangle ABC where A, B and C are the points $(-2, 2)$, $(2, 4)$ and $(7, -1)$, respectively

13.4 The circle

Mental task

What is a circle? What are the main parts of a circle that you can recall? Why are they significant?

In fact **the definition** of a circle is: the set of all points on a plane that are at a fixed distance from a centre.

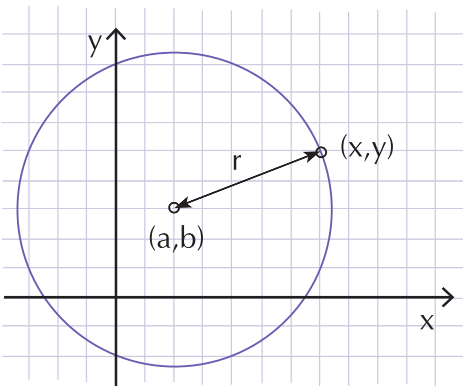


Fig 13.3

Let us put that centre at (a, b). So the circle is all the points (x, y) that are "r" away from the centre (a, b). Now we can work out exactly where all those points are.

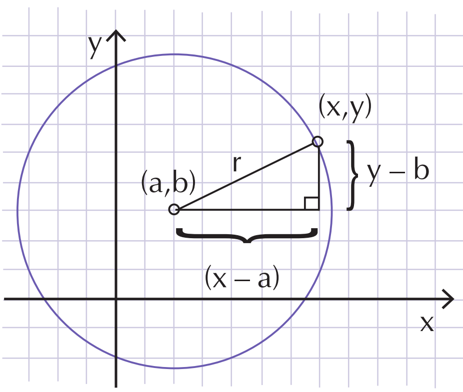


Fig 13.4

We make a right-angled triangle (Figure 13.4), and then use Pythagoras **($a^2 + b^2 = c^2$):**

$$(x - a)^2 + (y - b)^2 = r^2$$

And that is the **standard form** for the equation of a circle.

We can see all the important information at a glance: the centre **(a, b)** and the radius **r**.

But you may see a circle equation and not know it, because it is not in the standard form.

As an example, let us put some values to a, b and r and then expand it:

Start with:	$(x–a)^2 + (y–b)^2 = r^2$
Set (for example) a = 1, b = 2, r = 3:	$(x–1)^2 + (y-2)^2 = 3^2$
Expand:	$x^2 – 2x + 1 + y^2 - 4y + 4 = 9$
Gather like terms:	$x^2 + y^2 – 2x – 4y + 1 + 4 – 9 = 0$
And we end up with this:	$x^2 + y^2 – 2x – 4y – 4 = 0$

In fact we can write it in **general form** by putting constants instead of the numbers:

$x^2 + y^2 + Ax + By + C = 0$

Going from general form to standard form

Imagine we have an equation in general form:

$x^2 + y^2 – 2x – 4y – 4 = 0$

How can we get it into **standard form** like $(x – a)^2 + (y – b)^2 = r^2$?

The answer is to complete the square for x **and** for y:

Start with:	$x^2 + y^2 – 2x – 4y – 4 = 0$
Put xs and ys together on left:	$(x^2 – 2x) + (y^2 – 4y) = 4$
Simplify:	$(x^2 – 2x + (–1)^2) + (y^2 – 4y + (–2)^2) = 4 + (–1)^2 + (–2)^2$
Simplify:	$(x^2 – 2x + 1) + (y^2 – 4y + 4) = 9$
Finally:	$(x – 1)^2 + (y – 2)^2 = 3^2$

Unit circle

If we place the circle centre at (0,0) and set the radius to 1 we get:

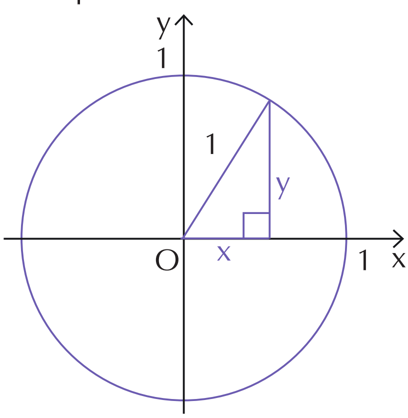


Fig 13.5

$$(x – a)^2 + (y – b)^2 = r^2$$
$$(x – 0)^2 + (y – 0)^2 = r^2$$
$$x^2 + y^2 = r^2$$

Generally, the circle is defined as the locus of all points, P(x, y), which are equidistant from some given point C (a, b). Suppose that the distance of the point P, from the given point C of coordinates (a, b), then

$(CP)^2 = (x – a)^2 + (y – b)^2 = r^2$

Thus the required locus is

$(x – a)^2 + (y – b)^2 = r^2$,

where r is the radius of the circle and the point C (a, b) is its centre.

If the coordinates of the point C are (0, 0), i.e. C is at the origin, then the equation becomes $x^2 + y^2 = r^2$, thus circle with centre (0, 0) and radius r has equation $x^2 + y^2 = r^2$ and circle with centre (a, b) and radius r has equation

$(x – a)^2 + (y – b)^2 = r^2$

Example 13.15

Find the centre and the radius of the circle with equation:

a) $(x – 2)^2 + y^2 = 25$

b) $x^2 + y^2 – 4x – 2y = 4$

Solution

$(x – 2)^2 + y^2 = 5^2$

by comparing this equation with $(x – a)^2 + (y – b)^2 = r^2$

a) the centre is (2, 0) and the radius is 5 units.

b) $x^2 + y^2 – 4x – 2y = 4$, which can be rearranged as

$(x^2 – 4x + 4) + (y^2 – 2y + 1) = 4 + 4 + 1$ i.e. $(x – 2)^2 + (y – 1)^2 = 3^2$;

comparing this equation with $(x – a)^2 + (y – b)^2 = r^2$, the centre is (2, 1) and the radius is 3 units.

Note: the general equation of circle, $(x – a)^2 + (y – b)^2 = r^2$ and may be written as $x^2 + y^2 – 2ax – 2by + (a^2 + b^2 – r^2) = 0$

Which is written as $x^2 + y^2 – Ax – By + C = 0$

Intersecting a line and a circle

Consider a straight line $y = mx + c$ and a circle $(x – a)^2 + (y – b)^2 = r^2$.

There are three possible situations:

1. The line cuts the circle in two distinct places, i.e. part of the line is a chord of the circle.
2. The line touches the circle, i.e. the line is a tangent to the circle.
3. The line neither cuts nor touches the circle.

Example 13.16

Show that part of the line $3y = x + 5$ is a chord of the circle $x^2 + y^2 - 6x - 2y - 15 = 0$ and find

a) the points of intersection

b) the length of the chord.

Solution

Substituting $x = 3y - 5$ into $x^2 + y^2 - 6x - 2y - 15 = 0$ gives, on simplification

$y^2 - 5y + 4 = 0$

$(y - 4)(y - 1) = 0$ then $y = 4$ or $y = 1$

If $y = 4$ then $x = 7$ and if $y = 1$ then $x = -2x$.

Therefore the line cuts the circle in 2 distinct points (7, 4) and (–2, 1).

The length of the cord is the distance between the point (7, 4) and (–2, 1) and is:

$\sqrt{[(7 - (-2)]^2 + (4 - 1)^2} = \sqrt{(18 + 9)} = \sqrt{90} = 3\sqrt{10}$ units.

Circle through three given points

Three non-collinear points define a circle, i.e. there is one, and only one circle which can be drawn through three non-collinear points. The equation of any circle may be written as $x^2 + y^2 + 2fx + 2gy + c = 0$ where f, g and c are constants. Thus, if we are given the coordinates of three points on the circumference of a circle, we can substitute these values of x and y into the equation of the circle and obtain three equations which can be solved simultaneously to find the constants g, f and c.

Example 13.17

Find the equation of the circle passing through the points (0, 1), (4, 3) and (1, -1).

Solution

Suppose the equation of the circle is $x^2 + y^2 + 2fx + 2gy + c = 0$.

Substituting the coordinates of each of the three points into this equation gives:

$1 + 2g + c = 0$ .... (1)

$25 + 8f + 6g + c = 0$ .... (2)

$2 + 2f - 2g + c = 0$ .... (3)

Multiplying equation (3) by 4 and the subtracting from equation (2), gives

$17 + 14g - 3c = 0$ .... (4)

Multiplying equation (1) by (3) and adding to equation (4), gives

$20 + 20g = 0$ or $g = -1$

From equation (1), $c = 1$ and from equation (3) $2f = -1 -2 -2 = -5$ $f = -\frac{5}{2}$

The equation of the circle which passes through (0, 1), (4, 3) and (1, –1) is

$x^2 + y^2 - 5x - 2y + 1 = 0$

Application activity 13.5

1. The equation of a circle in standard form is $(x + 11)^2 + (y - 9)^2 = 16$. What is the equation of the circle in general form?
2. A circle has centre (–7, 11) and the point (4, –3) lies on the circumference of the circle. What is the equation of the circle in standard form?
3. What is the radius of the circle $(x + 2)^2 + (y - 4)^2 = 36$?
4. What is the centre of the circle $x^2 + y^2 + 8x - 12y + 27 = 0$?
5. Given the Cartesian equation of a circle in the form $x^2 + y^2 + 4x + 6y = 12$, find by completing the squares, the centre coordinates of the circle and the radius. Hence sketch the graph of the circle, taking care to label the x-axis, the y-axis and the origin.

Summary

1. If A (x_1,y_1) and B(x_2, y_2) are two points, then the length of line joining points A and B. i.e the distance is

   $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ i.e. $d(AB) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
2. In 2D, a line can be defined by an equation in slope–intercept form, a vector equation, parametric equations, or a Cartesian equation (scalar equation).
3. The **vector equation** of a line is of the form $r = a + \lambda m$ where a is the position vector of a point on the line, m is parallel to the line and λ is a scalar.
4. If the acute angle between two lines is θ, then

   $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$
5. A circle is defined as the **locus of all points, P(x, y),** which are equidistant from some given point C (a, b).
6. There is one, and only one circle which can be drawn through three non-collinear points.
7. The equation of any circle may be written as

   $x^2 + y^2 + 2fx + 2fy + c = 0$, where f, g and c are constants.

Topic area: Statistics and probability

Sub-topic area: Descriptive statistics

Unit 14

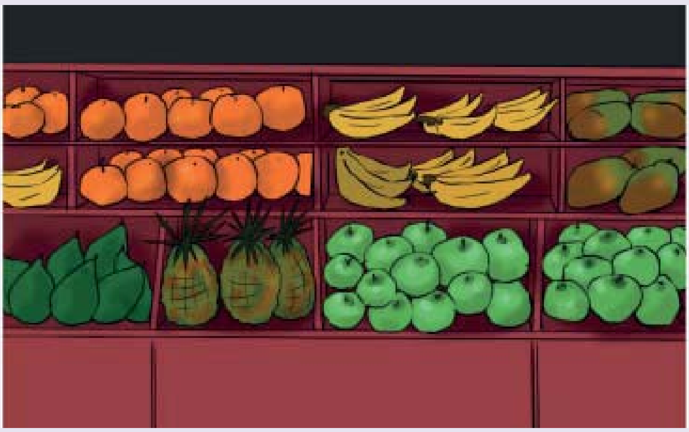
Measures of dispersion

Key unit competence

Extend understanding, analysis and interpretation of data arising from problems and questions in daily life to include the standard deviation.

14.0 Introductory activity

1. During 6 consecutive days, a fruit-seller has recorded the number of fruits sold per type.



The table below shows the types and the number of sold fruits in one week.

Type of fruit	A (Banana)	B (Orange)	C (Pineapple)	D (Avocado)	E (Mango)	F (apple)
Number of fruits sold	1100	962	1080	1200	884	900

a) Which type of fruits with the highest number of fruits that was sold?
b) Which type of fruits had the least number of fruits sold?
c) What was the total number of fruits sold that week?
d) Find out the average number of fruits sold per day.

14.1 Introduction

In Junior Secondary, you were introduced to statistics. You learnt about measures of central tendencies. In this unit, we shall learn about measures of dispersion.

Activity 14.1

Discuss the meaning of measures of central tendencies. What are they? Where can we apply them?

Definitions

A measure of central tendency; also called average, is values about which the distribution of data is approximately balanced. There are three types of measure of central tendency namely the mean, the median and the mode.

Mean: is the sum of data values divided by the number of values in the data

Mode: is the value that occurs most often in the data.

Median: is the middle value when the data is arranged in order of magnitude

The mean

The mean value of a set of data is the sum of all the items in the set of data divided by the number of items.

For discrete raw data mean = $\frac{\text{sum of items}}{\text{number of items}}$ i.e. $\bar{x} = \frac{\Sigma x}{n}$ where n is the number of items.

For example the mean of the numbers 8,10,11,13,15,16,19 and 22 is given by

$$\bar{x} = \frac{8 + 10 + 11 + 13 + 15 + 16 + 19 + 22}{8} = \frac{144}{8} = 14.25$$

For data in ungrouped frequency distribution

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

Example 14.1

The marks of 20 students in a mathematics test were recorded as follows:

Marks (x)	40	51	56	62	70	75	78
Frequency(f)	2	1	3	5	4	3	2

Find the mean.

Solution

x	f	fx
40	2	80
51	1	51
56	3	168
62	5	310
70	4	280
75	3	225
78	2	156
	Σ f = 20	Σ fx = 1270

Σ f = 20 and Σ fx = 1270

$\therefore$ mean = $\overline{x} = \frac{\Sigma fx}{f} = \frac{1270}{20} = 63.5$

The median

The median of data is the middle value when all values are arranged in order of the size.

When the number of the items is odd then the median is the item in the middle. If and when the number of items is even, the median is the mean of the two numbers in the middle.

Example 14.2

Find the median of :

a) 6 2 8 12 3 5 20 15 3

b) 9 4 5 6 8 10 2 7

Solution

a) Arranging the numbers in ascending order we have 2 3 3 5 6 8 12 15 20. These are 9 numbers, and the fifth is the middle of them thus the median is that number. Alternative formula:

$Median = x_{\frac{n+1}{2}}$ where n = number of items

So for our case we have the median = 6

b) Arranging the numbers in ascending order we have 2 4 5 6 7 8 9 10. They are 8 numbers. The median is the mean of the two middle numbers or simply the 4th and the 5th number.

We have $\frac{6+7}{2} = 6.5$

In general we have the formula $Median = \frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2}$ 1 or simply the $(\frac{n+1}{2})^{th}$ value.

Example 14.3

The table below shows the marks of 62 students in a test. Find the median.

marks	Number of students
40	2
41	4
42	6
43	9
44	10
45	12
46	8
47	7
48	2
49	1
50	1

Solution

Using the cumulative frequencies

Marks	Cummulative number
40	2
≤41	6
≤42	12
≤43	21
≤44	31
≤45	43
≤46	51
≤47	58
≤48	60
≤49	61
≤50	62

The total number of students is 62. In this case n = 62. Thus the median value is $\left(\frac{n+1}{2}\right)^{th}$ value = $\left(\frac{62+1}{2}\right)^{th}$ = 31.5^{th} value.

The median mark is the 31.5^{th} value which is the mean of 31 and 32. Starting with the lowest mark of 40 and move up the frequencies until you reach the 31^{st} and the 32^{nd} share in the distribution.

2+4+6+9+10=31 so the 31st student obtained 44 marks and the 32nd student obtained 45 marks

So the median is $\frac{44 + 45}{2}$ = 44.5. Alternatively. by using the formula

Median = $\frac{x_{\frac{n}{2}} + x_{\frac{n}{2}+1}}{2} = \frac{x_{31} + x_{32}}{2} = \frac{44 + 45}{2}$ = 44.5

The mode

The mode of a set of data is the value of the higher frequency in the distribution of marks

In Example 14.3 the mode is 45 because it has the highest frequency, 12.

Grouped data

Grouped data is commonly used in continuous distribution data that takes any value in a given range is called **continuous data**.

Such data has values which are only approximations, such as height, weight, mass, time, age and temperature.

The frequency distribution below shows the lengths of metal rods measured to the nearest millimetre.

Length	20–24	25–29	30–34	35–39	40–44
Frequency	5	9	13	11	6

The interval 25–29 means that the length is equal to or greater than 24.5 mm and than 29.5 mm, written as 24.5 mm< l <29.5 mm.

Class boundary	**Class width**
19.5 – 24.5	5
24.5 – 29.5	5
29.5 – 34.5	5
34.5 – 39.5	5
39.5 – 44.5	5

The mean of grouped data

In order to find the mean of the grouped data:

I Find the mid-point of each interval.

II Multiply the mid-point (x) by the frequency (f) of each interval to find f.x.

III Find the sum denoted by Σfx and divide by Σf to obtain the mean.

Example 14.4

Find the mean of the following distribution

Mass (kg)	10–19	20–29	30–39	40–49	50–59	60–69
Frequency	5	7	15	12	8	3

Solution

Mass(kg)	**Mid-point**	**Frequency**	**f.x**
10–19	14.5	5	72.5
20–29	24.5	7	171.5
30–39	34.5	15	517.5
40–49	44.5	12	534.5
50–59	54.5	8	436.0
60–69	64.5	3	193.5

$\Sigma f = 50$ and $\Sigma fx = 1925$ and so the Mean $\frac{\Sigma fx}{\Sigma f} = \frac{1925}{50} = 38.5$kg.

The mean from an assumed mean

When data is grouped in classes of equal width, we use the assumed mean in order to reduce the numerical size of the value of n.

Determine the mid-point of each class interval and the classes of the central value of x which is usually the modal value. This value is referred to as an **assumed value** or **working mean**.

Subtract the assumed mean from each value of x and where necessary divide the difference so obtained by the class width. This process is a new set of values, say y, that is:

$$y = \frac{x - \text{assumed mean}}{\text{class width}}$$

The mean of y denoted as y is given by the equation $\bar{y} = \frac{\Sigma fy}{\Sigma f}$ and the mean of x denoted by $\bar{x}$ is given by the formula $\bar{x}$ = assumed mean + $\bar{y}$.class width

Example 14.5

Calculate the mean of the following data using the assumed mean

Mass(kg)	Frequency
10–19	3
20–29	7
30–39	12
40–49	18
50–59	12
60–69	6

Let assumed mean be 44.5

Solution

Class	mid-point	f	$y = \frac{x - 44.5}{10}$	f. y
10–19	14.5	3	–3	–9
20–29	24.5	7	–2	–14
30–39	34.5	12	–1	–12
40–49	44.5	18	0	0
50–59	54.5	12	1	12
60–69	64.5	6	2	12

$\sum fy = -11$

$\sum f = 58$, $\bar{y} = \frac{\Sigma fy}{\Sigma f} = \frac{-11}{58} = -0.19$

$\bar{x} = 44.5 + (-0.19 \times 10) = 42.6$ The mean of the given data is 42.6.

14.2 Measures of dispersion

Activity 14.2

In groups of three, research on the meaning and types of measures of dispersion. Why are they useful? Discuss your findings with the rest of the class.

A measure of dispersion is the degrees of spread of observation in data. The common measure of dispersion are range inter-quartiles, range and the standard deviation (the square root of the variance)

Range

The range is defined as the difference between the largest value in the set of data and the smallest value in the set of data, $X_L - X_S$

Example 14.6

What is the range of the following data?

4 8 1 6 6 2 9 3 6 9

Solution

The largest score (X_L) is 9; the smallest score (X_S) is 1;
the range is $X_L - X_S = 9 - 1 = 8$

The range is rarely used in scientific work as it is fairly insensitive.

It depends on only two scores in the set of data, X_L and X_S

Two very different sets of data can have the same range:

For example 1 1 1 1 9 and 1 3 5 7 9.

Inter-quartile range

The other measure of dispersion is the difference between two percentiles, usually the 25th and the 75th percentiles. For numerical data arranged in ascending order, the quartiles are values derived from the data which divide the data into four equal parts. If there are n observations, the first quartile (or lower quartile) Q1 is the $\frac{1}{4}(n+1)^{th}$ data, the second quartile Q2 (the median) is the $\frac{1}{2}(n+1)^{th}$ data and the third quartile (or upper quartile) Q3 is the $\frac{3}{4}(n+1)^{th}$ data. When the $\frac{1}{4}(n+1)^{th}$ is not a whole number, it is sometimes thought necessary to take the (weighted) average of two observations, as is done for the median. However, unless n is very small, an observation that is nearest will normally suffice.

The inter-quartile range is the central 50% of a distribution when it is arranged in order of size. It is given by the formula $Q_3 - Q_1$ where Q_1 is the lower quartile and Q_3 the upper quartile.

Semi inter-quartile range

The semi-interquartile range (or SIR) is defined as the difference of the first and third quartiles divided by two

The first quartile is the 25th percentile

The third quartile is the 75th percentile

$$SIR = \frac{(Q_3 - Q_1)}{2}$$

The semi inter-quartile range also called the quartile deviation is the half inter-quartile. That is $\frac{Q_3 - Q1}{2}$.

It is not always true that the quartile deviation is $Q_3 - Q_1$ or $Q_2 - Q_1$

Example 14.7

For the data below

5, 3, 7, 6, 3, 4, 2, 7, 5, 6, 5, 4, 9, 8, 3, 6, 5

Find the

(a) first (lower) quartile

(b) second quartile (the median)

(c) third (upper) quartile

(d) inter-quartile range (IQR)

(e) quartile deviation.

Solution

The ordered data set is:

2, 3, 3, 3, 4, 4, 5, 5, 5, 5, 6, 6, 6, 7, 7, 8, 9

There are 17 data, so n=17

a) The first quartile is $\frac{1}{4}(17+1)^{th}$ data = $(4.5)^{th}$ data= $\frac{3-4}{2}$ = 3.5

b) The second quartile is $\frac{1}{2}(17+1)^{th}$ data = 9^{th} data = 5

c) The upper quartile is $\frac{3}{4}(17+1)^{th}$ data = $(13.5)^{th}$ data $= \frac{6-7}{2}$ =6.5

d) IQR = $Q_3 - Q_1$= 6.5 – 3.5 = 3

e) The quartile deviation is $\frac{Q_3 - Q1}{2} = \frac{6.5 - 3.5}{2} = \frac{3}{2}$ =1.5

Variability of data

Each of these sets of numbers has a mean of 7 but the spread of each set is different:

(a) 7, 7, 7, 7, 7

(b) 4, 6, 6.5, 7.2, 11.3

(c) –193, –46, 28, 69, 177

There is no variability in set (a), but the numbers in set (c) are obviously much more spread out than those in set (b).

There are various ways of measuring the variability or spread of a distribution, two of which are described here.

The range is based entirely on the extreme values of the distribution.

In (a) the range = 7– 7 = 0

In (b) the range =11.3 – 4=7.3

In (c) the range =177 – (–193) =370

Note that there are also ranges based on particular observations within the data and these are **percentile** and **quartile ranges**.

The standard deviation and the variance

The standard deviation, s , is a very important and useful measure of spread. It gives a measure of the deviations of the readings from the mean, $\overline{x}$. It is calculated using all the values in the distribution.

To calculate standard deviation, s:

(i) For each reading of x calculate $x - \overline{x}$, its deviation from the mean

(ii) Square this deviation to give $(x - \overline{x})^2$ and note that, irrespective of whether the deviation was positive or negative, this is now positive

(iii) Find $\Sigma(x - \overline{x})^2$, the sum of all these values,

(iv) Find the average by dividing the sum by n, the number of readings; this gives $\frac{\sum(x-\overline{x})^2}{n}$ and is known as **variance**

(v) Finally, take the positive square root of the variance to obtain the standard deviation, s.

The standard deviation, s, of a set of n numbers, with mean $\overline{x}$, is given by $s = \sqrt{\frac{\Sigma(x - \overline{x})^2}{n}}$. Each of the following three sets of numbers has mean 7, i.e. $\overline{x} = 7$

(a) For the set 7, 7, 7, 7, 7 since $x - \overline{x}$ = for every reading, s = 0, indicating that there is no deviation from the mean.

(b) For the set 4, 6, 6.5, 7.2, 11.3

$$\sum (x - \overline{x})^2 = (4 - 7)^2 + (6 - 7)^2 + (6.5 - 7)^2 + (7.2 - 7)^2 + (11.3 - 7)^2 = 28.78$$

$$s = \sqrt{\frac{\Sigma(x - \overline{x})^2}{n}} = \sqrt{\frac{28.78}{5}}$$

(c) For the set –193, –46, 28, 69, 177

$$(x - \overline{x})^2 = (-193 - 7)^2 + (-46 - 7)^2 + (28 - 7)^2 + (69 - 7)^2 + (117 - 7)^2$$

$$= 75994$$

$$\sqrt{\frac{\Sigma(x - \overline{x})^2}{n}} = \sqrt{\frac{75994}{5}} = 123.3$$

Notice that set (c) has a much higher standard deviation than set (b), confirming that it is much more spread about the mean

Remember that

Standard deviation = $\sqrt{\text{variance}}$

Variance = (standard deviation)^2

Note

The standard deviation gives an indication of the lowest and highest values of the data. In most distributions, the bulk of distribution lies within two standard deviations of the mean, i.e. within the interval $\overline{x} \pm 2s$ or $(\overline{x} - 2s, \overline{x} + 2s)$. This helps to give an idea of the spread of the data.

The units of standard deviation are the same as the units of the data.

Standard deviations are useful when comparing sets of data; the higher the standard deviation, the greater the variability in the data.

Example 14.7

Two machines A and B are used to pack biscuits. A random sample of ten packets was taken from each machine and the same mass of each packet was measured to the nearest gram and noted. Find the standard deviations of the masses of the packets taken in the sample from each machine. Comment on your answer.

Machine A (mass in g)	196, 198, 198, 199, 200, 200, 201, 201, 202, 205
Machine B (mass in g)	192, 194, 195, 198, 200, 201, 203, 204, 206, 207

Solution

Machine A: $\overline{x} = \frac{\Sigma x}{n} = \frac{2000}{10} = 200$. Machine B: $\overline{x} = \frac{\Sigma x}{n} = \frac{2000}{10} = 200$

Since the mean mass for each machine is 200, $x - \overline{x} = x - 200$

To calculate standard deviation, s; put the data in a table:

Machine A		
x	x – 200	$(x - 200)^2$
196	–4	16
198	–2	4
198	–2	4
199	–1	1
200	0	0
200	0	0
201	1	1
201	1	1
202	2	4
205	5	25
		56

$$s^2 = \frac{\Sigma(x - 200)^2}{10} = 5.6$$

$$s = \sqrt{5.6} = 2.37$$

Machine A: s.d = 2.37g

Machine B		
x	x – 200	$(x - 200)^2$
192	–8	64
194	–6	36
195	–5	25
198	–2	4
200	0	0
201	1	1
203	3	9
204	4	16
206	6	36
207	7	49
		240

$s^2 = \frac{\Sigma(x - 200)^2}{10} = 24$

$s = \sqrt{24} = 4.980$ g

Machine B: s.d = 4.90 g

Machine A has less variation, indicating that it is more reliable than machine B.

Alternative form of the formula for standard deviation

The formula given above is sometimes difficult to use especially when $\overline{x}$ is not an integer; so an alternative form is often used. This is derived as follows:

$s^2 = \frac{1}{n}\Sigma(x - \overline{x})^2 = \frac{1}{n}\Sigma(x^2 - 2\overline{x}\,x + \overline{x}^2) = \frac{1}{n}(\Sigma x^2 - 2\overline{x}\,\Sigma x + \Sigma\overline{x}^2)$

since, $\frac{\Sigma x}{n} = \overline{x}$

$s^2 = \frac{\Sigma x^2}{n} - 2\overline{x}\frac{\Sigma x}{n} + \frac{n\overline{x}^2}{n} = \frac{\Sigma x^2}{n} - 2\overline{x}(\overline{x}) + \overline{x}^2 = \frac{\Sigma x^2}{n} - 2\overline{x}^2 + \overline{x}^2 = \frac{\Sigma x^2}{n} - \overline{x}^2$

$s^2 = \frac{\Sigma x^2}{n} - \overline{x}^2$

$s = \sqrt{\frac{\Sigma x^2}{n} - \overline{x}^2}$

Application activity 14.1

1. For each of the following sets of numbers, calculate the mean and the standard deviation.

   (a) 2, 4, 5, 6, 8

   (b) 6, 8, 9, 11

   (c) 11, 14, 17, 23, 29

2. For each of the following sets of numbers, calculate the mean and standard deviation using one of the methods of the formula for the standard deviation.

   (a) 5, 13, 7, 9, 16, 15

   (b) 4.6, 2.7, 3.1, 0.5, 6.2

   (c) 200, 203, 206, 207, 209

14.3 Coefficient of variation

The coefficient of variation (CV) is defined as the ratio of the standard deviation σ to the mean μ.

$$CV = \frac{\sigma}{\mu}$$

It shows the extent of variability in relation to the mean of the population. The coefficient of variation should be computed only for data measured on a ratio scale, as these are the measurements that can only take non-negative values.

The coefficient of variation (C.V) unlike the previous measures we have studied is a relative measure of dispersion. It is expressed as a percentage rather than in terms of the unit of the particular data. It is useful when comparing the variable of two or more batches of data. Those are expressed in different units of measurement.

$C.V = \frac{\delta}{\overline{x}} \times 100$ where δ is the standard deviation and $\overline{x}$ is the mean.

For example, given that $\delta = 6.26$ and $\overline{x} = 20$ then $CV = \frac{6.26}{20} \times 100 = 31.3\%$

That is for this sample the relative size of the average spread around the mean is 31.3%. The C.V is also very useful when comparing two or more sets of data which are measured in the same units.

The following are more examples.

A data set of [100, 100, 100] has constant values. Its standard deviation is 0 and average is 100:

The CV is $100\% \times \frac{0}{100} = 0\%$

A data set of [90, 100, 110] has more variability. Its standard deviation is 8.16 and its average is 100:

The CV is $100\% \times \frac{8.16}{100} = 8.16\%$

A data set of [1, 5, 6, 8, 10, 40, 65, 88] has more variability again. Its standard deviation is 30.78 and its average is 27.875:

The CV is $100\% \times \frac{30.78}{27.875} = 110.4\%$

14.4 Application

Activity 14.3

Carry out research to find out real life applications on measures of dispersion. Discuss your findings with the rest of the class.

Example 14.9

Sona, Karina, Omar, Mustafa and Amie earned scores of 6, 7, 3, 7 and 2 respectively on a standardized test.

Find the mean deviation and standard deviation of their scores.

Solution:

Mean deviation: We must first find the mean of the data set. The mean is

$\frac{6 + 7 + 3 + 7 + 2}{5}$ = 5. Then the mean deviation is calculated by

$$\frac{|6-5|+|7-5|+|3-5|+|7-5|+|2-5|}{5} = \frac{1+2+2+2+3}{5} = \frac{10}{5} = 2.$$

The mean deviation is 2.

Standard deviation is

$$= \sqrt{\frac{[(6-5)^2 + (7-5)^2 + (3-5)^2 + (7-5)^2 + (2-5)^2]}{5}}$$

$$= \frac{1+4+4+4+9}{5}$$

$$= \frac{\sqrt{22}}{5} \approx 0.938$$

The standard deviation is approximately 0.938

Application activity 14.2

1. The marks of a class in a test are as follows

   52, 45, 25, 75, 63, 86, 72, 85, 55, 65, 70, 82, 90, 48, 68, 86, 65, 64, 78, 75, 32, 42. Find the inter-quartile range.

2. Find the standard deviation of the data set 5, 10, 15, 20, 25, 30, 35, 40, 45, 50.

3. Calculate the variance and the standard deviation for the following values: 1, 3, 5, 6, 6, 8, 9, and 10.

4. Ten different teams played football during one season. At the end of the season the top goal scorers from each team scored the following number of goals:

   10, 5, 18, 2, X, 4, 10, 15, 11, 4

If the mean number of goals scored is 9, what is the:

a) value of X?

b) mode?

c) median?

d) range?

e) mean deviation?

f) standard deviation?

g) 50^{th} percentile?

h) percentile of the goal scorer with 11 goals scored?

Summary

1. Mean: is the sum of data values divided by the number of values in the data
2. Mode: is the value that occurs most often in the data.
3. Median: is the middle value when the data is arranged in order of magnitude
4. When data are grouped in classes of equal width, we use the **assumed mean** in order to reduce the numerical size of the value of n.
5. A **measure of dispersion** is the degree of spread of observation in data. The common measures of dispersion are inter-quartiles, range and the standard deviation (the square root of the variance).
6. Range: is the numerical difference between the largest value and the least value of data
7. The **inter-quartile range** is the central 50% of a distribution when it is arranged in order of size.
8. The **coefficient of variation** (C.V) is a relative measure of dispersion. It is expressed as a percentage rather than in terms of the unit of the particular data. It is useful when comparing the variable of two or more batches of data.
   C.V. = $\frac{\delta}{\overline{X}}$ x 100 where δ is the standard deviation and $\overline{x}$ is the mean.

Topic area: Statistics and probability

Sub-topic area: Combinatorial analysis and probability

Unit 15

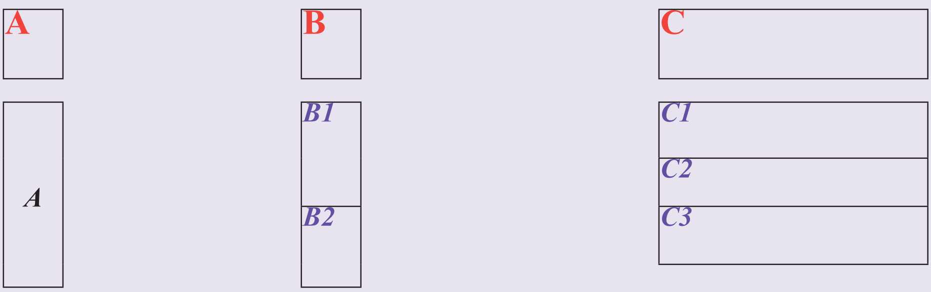
Combinatorics

Key unit competence

Use combinations and permutations to determine the number of ways a random experiment occurs.

15.0 Introductory activity

There are 2 roads joining A and B and 3 roads joining B and C. Write down different roads from A to C via B. How many are they? Discuss the different methods you can apply to find the correct answer.



15.1 Counting techniques

Combinatorics , also known as combinatorial analysis, is the area of mathematics concerned with counting strategies to calculate the ways in which objects can be arranged to satisfy given conditions.

Venn diagrams

This is a way of representing sets in a closed curve. They are usually circular or oval shaped. The universal set, written U or ε, is represented by a rectangle and is a set containing all the sub-sets considered in the problem. In the probability theory, the universal set is called the **sample space** and its subsets are called the **events**.

The complement of a set say A is the set written A′ or $\bar{A}$ of an element in the universal set but not in the set A i.e. A′ = U – A and we can write A ∪ A′ = U.

n(A) or |A| read as the number of elements in set A. This is also called the **cardinal number** of A.

Intersection of sets A and B is denoted by A ∩ B. This represents the set of all elements common to both A and B.

Union of set A and B is denoted by A ∪ B. This represents the set of all elements belonging to A or B or both A and B.

Disjoint sets are sets with no elements in common. If A and B are disjoint sets, then A ∩ B = ∅ . Also A and B are said to be **mutually exclusive**.

Example 15.1

If U = {a, b, c, d, e} and A = {a, b, c} then A′ = {d, e }

Application activity 15.1

1. Let U = {1, 2, 3, 4, 5, 6, 7}, A = {1, 3, 5, 7}, B = {4, 5, 6, 7} and C = {2, 3, 5, 7}. List the elements in each of the following sets:

   a) A ∪ B b) A ∩ C c) B – C

   d) A ∪ B′ e) C′ – A′ f) A′ ∩ C

2. If n ∈ $\mathbb{Z}^+$, list the elements of each of the following sets:

   a) $\{n|2< n < 7\}$ b) $\{n^2|1\le n \le 5\}$ c) $\{n|1 \le n^2\le 40\}$;

   d) {n|n divides 143} e) $\{n|n^2 = 5 – 4n\}$ f) $\{n|2n^2 = 5n - 3\}$.

3. In each of the following, draw a Venn diagram showing the universal set U and the sets A, B and C. Shade the region represented:

   a) A ∪ B ∪ Cb) A ∩ B ∩ C c) A′ ∩ B

   d) (A ∪ B) ∩ C′ e) B′ ∪ C′ f) A′ ∩ B′ ∩ C′

Tree diagrams

A tree diagram is a diagram with a structure of branching connecting lines representing a relationship. It can be used to find the number of possible outcomes of experiments where each experiment occurs in a finite number of ways. For example, when you toss a coin, the outcome is either head or tail. A second toss would also give head or tail. We represent this as:

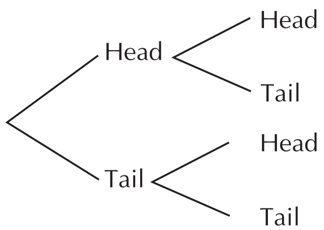


Fig 15.2

Example 15.2

Akimana and Bisangwa play a game of tennis in which the first person to win two sets becomes the winner. In how many different ways can this be done?

Solution

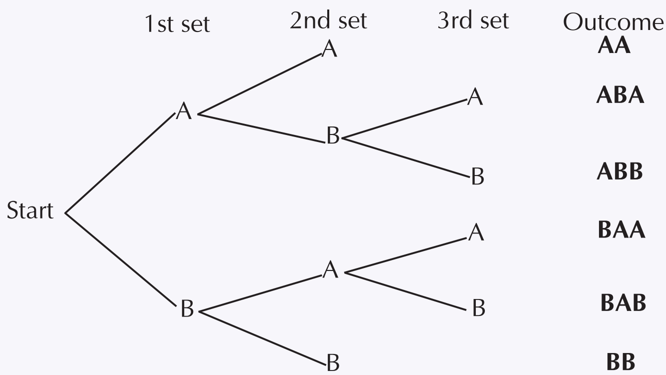


Fig 15.2

Thus the number of possible outcomes of the game is 6.

Contingency table

This is a method of presenting the frequencies of the outcomes from an experiment in which the observations in the sample are categorised according to two criteria. Each cell of the table gives the number of occurrences for a particular combination of categories. An example of contingency table in which individuals are categorised by gender and performance is shown below.

	Male	**Female**	**Total**
Pass	32	43	75
Fail	8	15	23
	40	58	98

A final column giving the row sums and a final row giving the column sums may be added. Then the sum of the final column and the sum of the final row both equal the number in the sample.

If the sample is categorized according to three or more criteria, the information can be presented similarly in a number of such tables.

Multiplication rule

This is also known as product rule or product principle or multiplication principle.

Suppose that an experiment (operation) is to be performed in two successive ways. If there are n_1 different ways of performing the first operation and for each of these there are n_2 different ways of performing a second independent operation, then the two operations in succession can be performed in $n_1 \times n_2$ different ways.

More generally, if the operation is composed of k successive steps which may be performed in $n_1, n_2, \ldots, n_k$ distinct ways, respectively, then the operation may be performed in $n_1 \times n_2 \times \ldots \times n_k$ distinct ways.

Example 15.3

There are 3 different paths from Nizeyimana's to Mukaneza's and 2 different paths from Mukaneza's' to Uwimana's. How many different pathways could be taken from Nizeyimana's to Uwimana's via Mukaneza's?

Solution

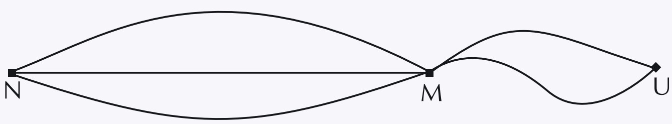


Fig 15.3

The number of different pathways is 3 x 2 = 6

Example 15.4

Habimana has 4 shirts, 3 pair of trousers and 2 pairs of shoes. He chooses a shirt, a pair of trousers and a pair of shoes to wear every day. Find the maximum number of days he does not need to repeat his clothing.

Solution

The number of all possible different clothing is 4 x 3 x 2 = 24. Therefore

The maximum number of days Habimana does not need to repeat his clothing is 24.

Example 15.5

Find how many ways the first three places (with no ties) can be filled in a race with 5 contestants.

Solution

The first place can be filled in 5 ways, since any contestant can come first. When the first place has been filled, there are 4 more contestants to chose from for the second place. Hence the second place can be filled in 5 x 4 ways. Finally, for each of these ways, the third place can be filled by any of the remaining 3 contestants, and the total number of ways is 5 x 4 x 3 = 60

15.2 Arrangements and permutations

Mental task

Imagine that you are a photographer. You want to take a photograph of a group of say 7 people. In how many different ways can they be arranged in a single row?

Factorial notation

The factorial notation of n integers denoted by n! is the product of the first consecutive integers and is read as 'factorial n'. Thus, $n! = n \times (n-1) \times (n-2) \times \ldots \times 4 \times 3 \times 2 \times 1$.

Thus, for example $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Note: $0! = 1$

Property: $\frac{n!}{r!} = n \times (n-1) \times (n-2) \times \ldots \times (r+1)$

Thus, for example $\frac{9!}{5!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 2}{5 \times 4 \times 3 \times 2 \times 1} = 9 \times 8 \times 7 \times 6$

Permutations

Activity 15.1

In how many different ways can a group of 3 students be arranged to sit in a row?

Share your findings with the rest of the class.

Arrangements without repetition

Suppose that two of the three different pictures, A, B and C are to be hung, in line, on a wall of three places. The pictures can be hung in different orders:

A B	A C
B A	B C
C A	C B

Each of these orders is a particular arrangement of the pictures and is called a permutation.

Thus, a permutation is an ordered arrangement of the items in a set.

Here there are 6 permutations, or $\frac{3!}{(3-2)!} = \frac{3!}{1!} = 6$

The number of arrangements of r objects, taken from n unlike objects, can be considered as the number of ways of filling r places in order by the n given objects and is given by

$p_r^n = \frac{n!}{(n-r)!}$

Note: A different notation can be used of permutation with no repetition:

$P_r^n = P(n, r) = {}_nP_r$

Condition

$n \geq r$

Example 15.6

Find the number of permutations of 6 objects taken 3 at time.

Solution

The first object can be put in any of 6 places; following this, the second object can be put in any of 5 remaining places; following this, the last object can be put in any of 4 remaining places. Thus, by the product principle, there are $6 \times 5 \times 4 = 120$ possible permutations of 6 objects taken 3 at time.

(i.e $P_3^6 = \frac{6!}{(6-3)!} = 120$).

Example 15.7

Find how many ways the first three places (with no ties) can be filled in a race with 15 contestants.

Solution

The first place can be filled in 15 ways, since any contestant can come first. When the first place has been filled, there are 14 more contestants to choose from for the second place. Hence the first two places can be filled in 15 x 14 ways. Finally, for each of these ways, the third place can be filled by any of the remaining 13 contestants, and the total number of ways is 15 x 14 x 13 = 2730. Alternatively, from the formula

$P_3^{15} = \frac{15!}{(15-3)!} = \frac{15!}{12!} = 15 \times 14 \times 13 = 2730$

Note: If n = r, this requires us to arrange n different objects in n different places and the number of different ways is:

$$P^n_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Example 15.8

How many different ways can 10 students be arranged in a class of 10 places?

Solution

The number of different ways is 10! = 3,628,800 ways.

Arrangements with repetition

If repetition is allowed, each place can be filled by the objects in n different ways. For example, to the six arrangements of a, b, c without repetitions (AB, AC, BA, BC, CA, and CB) are added the three with repetitions (AA, BB, and CC), for a total of 9, which is equal to 3^2. Thus, the number of arrangements of r objects, taken from n unlike objects where each of which may be repeated any number of times is n^r.

Example 15.9

Find how many three letter codes can be formed from an alphabet of 26 letters (assuming that letters can be repeated).

Solution

The first letter of the code may be any of the 26 in the alphabet. The same is true for the second letter and for the third letter. Therefore, the total number of codes is

$26 \times 26 \times 26 = (26)^3 = 1,757,626$

Example 15.10

Three schools have teams of six or more runners in a cross country race. In how many ways can the first six places be taken by the three schools, if there are no dead heats?

Solution

The first place can be taken by any one of the three schools. When the first runner has come in, the second place can be taken by any of the three schools. Continuing the argument for the third, fourth, fifth and sixth, the first six places may be taken by the three schools in $3^6 = 729$ ways.

Example 15.11

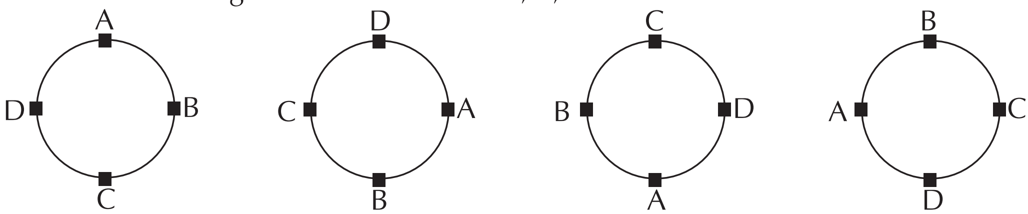
Find the number of permutations of 4 objects in a line (all taken at a time).

Solution

The first object can be arranged in any of the 4 different places. The second object can be arranged in three remaining places and the third object can be arranged in 2 remaining places and finally the last object can be arranged in 1 remaining place. Thus, by the product principle, the number of permutations of the 4 objects is 4 x 3 x 2 x 1 = 24 Therefore, the number of permutations of 4 object is 4!=24.

Arrangement in a circle

Observe the arrangement of four letters A, B, C and D on a circle as shown below.



We can say that for all the arrangements:

A is on the opposite side of C and its neighbours are B and D, and

B is on the opposite side of D and its neighbours are A and C.

Thus all of the four arrangements are the same.

In arranging 4 different objects on a circle, 4 are indistinguishable. Therefore the number of different ways of arranging 4 objects on a circle is $\frac{4!}{4} = 3!$

The number of arrangements of n unlike things in a circle will therefore be $(n-1)!$. In those cases where clockwise and anticlockwise arrangements are not considered to be different, this reduces to $\frac{1}{2}(n-1)!$.

Example 15.12

In how many different ways can 5 people be arranged in a circle?

Solution

The number of different ways of arranging 5 people is (5 – 1)! = 4! = 24.

Conditional arrangements

Sometimes arrangements may have certain restrictions which should be dealt with first.

Example 15.13

How many arrangements of the letters in the word GROUP start with a vowel?

Solution

Since we have two vowels, there are only 2 possibilities to choose the first letter. When the first has been chosen, the second can be any one of the remaining 4 letters (including the vowel not chosen), and 3 for the third, 2 for the fourth and 1 for the last. Therefore, the number of arrangements in the word GROUP starting with a vowel is $2 \times 4 \times 3 \times 2 \times 1 = 2(4!) = 48$

Example 15.14

Four boys and two girls are to sit in a row. The two girls, Mukaneza and Mukakarangwa insist on sitting together. In how many different ways can the six students be arranged?

Solution

As Mukaneza and Mukakarangwa must sit together we first treat them as one unit i.e. we have only five items to arrange, four boys and a pair of girls. This is done in 5! ways. In any one of these arrangements, the two girls can be arranged two ways among themselves i.e. 2!. Thus, the total number of arrangements is $2! \times 5! = 240$.

Arrangement with indistinguishable elements

In arranging objects, some sets may contain certain elements that are indistinguishable from each other. Below is how we find the number of arrangements.

The number of ways of arranging n objects in a line, of which p are alike, is $\frac{n!}{p!}$.

Example 15.15

How many different words of 5 letters (not necessarily sensible), can be formed from the letters of the words

(a) MATHS (b) POPPY

Solution

(a) All the letters are distinguishable, so the number of different words is $5!=\mathbf{120}$

(b) The 3 'p's are indistinguishable, so the number of different words is $\frac{5!}{3!} = 5 \times 4 = 20$.

In general,

The number of ways of arranging in a line n objects of which p of one type are alike, q of a second type are alike, r of a third type are alike, and so on, is $\frac{n!}{p!q!r!}$.

Example 15.16

In how many different ways can 4 identical red balls, 3 identical green balls and 2 yellow balls be arranged in a row?

Solution

The number of ways is $\frac{9!}{4! \times 3! \times 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 6 \times 2} = \frac{9 \times 8 \times 7 \times 6 \times 5}{12} = 1260$

Application activity 15.2

1. How many different ways can four people be arranged in a car of five seats if

   a) one of them can drive b) any one of them can drive.

2. Six people want to take a photo. Find how many different ways they can stand on a line.
3. How many different ways can 11 people be arranged at a round table?
4. How many 3-digit numbers can be made from the figures 1, 2, 3, 4, 5 when

   a) repetitions are allowed b) repetitions are not allowed?

5. Find the number of ways of arranging five different books in a row in a bookshelf.
6. There are ten teams in the local football competition. In how many ways can the first four places in the premiership table be filled?
7. In how many different ways can five identical blue balls, two identical red balls and a yellow ball be arranged in a row?

8. Find the number of arrangements of four different letters chosen from the word PROBLEM which
   (a) begin with a vowel (b) end with a consonant.
9. In how many ways can three books be distributed among ten people if
   (a) each person may receive any number of books
   (b) no person may be given more than one book
   (c) no person may be given more than 2 books?
10. (a) In how many ways can five men, four women and three children be arranged in a row so that the men sit together, the women sit together and the children sit together?
    (b) Repeat part (a) but this time if they sit around a circular table.

15.3 Combinations

Activity 15.2

Discuss in groups of 5: in how many ways can a committee of 5 students be chosen from a class of 30 students?

Share your findings with the rest of the class.

Definitions and properties

Combination is a selection of objects where the order is not taken into account - the order in which the objects are arranged is not important.

When considering the number of combinations of r objects, the order in which they are placed is not important.

Note that ABC, ACB, BAC, BCA, CAB, CBA are different arrangements, but they represent the same combination of letters.

Denoting the number of combinations of three letters from the seven letters A, B, C, D, E, F, G, by ${}^7C_3$ then ${}^7C_3 \xi 3! = {}^7P_3$

${}^7C_3 = {}^7P_3/3! = \frac{7!}{3!4!} = 35$

In general, the number of combinations of r objects from n unlike objects is

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Condition

$n \geq r$

Note: Other notations that can be used are ${}^{n}C_{r}$, C_r^n, ${}_{n}C_{r}$ or $C(n, r)$.

Some properties are

1) $\binom{n}{r} = \binom{n}{n-r}$

2) $\binom{n}{n} = \binom{n}{o} = 1$

3) $\binom{n}{1} = n$

There is a relationship between combinations and arrangements:

$$\binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

Example 15.17

How many different committees of 3 people can be chosen from a group of 12 people?

Solution

The number of committees is

$$\binom{12}{3} = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

Conditional combination

Sometimes we are given a condition that must be taken into account in a combination.

Example 15.18

A committee of 5 is to be chosen from 12 men and 8 women. In how many ways can this be done if there are to be 3 men and 2 women on the committee?

Solution

The number of ways of choosing the men is

$$\binom{12}{3} = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

The number of ways of choosing the women is

$$\binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{2! \times 6!} = \frac{8 \times 7}{2 \times 1} = 28$$

Therefore, the total number of ways of choosing the committee (given by the product rule) is 220 x 28 = 6160

Example 15.19

A group consists of 4 boys and 7 girls. In how many ways can a team of three be selected if it is to contain:

a) 1 boys and 2 girls

b) Girls only

c) Boys only

d) At least 2 boys

e) At least one member of each gender?

Solution

a) $\binom{4}{1} \times \binom{7}{2} = \frac{4!}{1! \times (4-1)!} \times \frac{7!}{2! \times (7-2)!} = \frac{4 \times 3!}{1! \times 3!} \times \frac{7 \times 6 \times 5!}{2! \times 5!} = 4 \times 21 = 84$

b) $\binom{7}{3} = \frac{7!}{3!(7-3)} = \frac{7 \times 6 \times 5 \times 4!}{3! \times 4!} = 35$

c) $\binom{4}{3} = \frac{4!}{3!(4-3)} \times \frac{4 \times 3!}{3! \times 1!} = 4$

d) $\binom{4}{2} \times \binom{7}{1} + \binom{4}{3} \times \binom{7}{0} = \frac{4!}{2! \times (4-2)!} \times \frac{7!}{1!(7-1)!} + 4 \times 1 = \frac{4 \times 3 \times 2!}{2! \times 2!} \times \frac{7 \times 6!}{1 \times 6!} + 4$

$= 6 \times 7 + 4 = 42 + 4 = 46$

e) $\binom{4}{1} \times \binom{7}{2} + \binom{4}{2} \times \binom{7}{1} = \frac{4!}{1! \times (4-1)!} \times \frac{7!}{2! \times (7-2)!} + \frac{4!}{2! \times (4-2)!} \times \frac{7!}{1! \times (7-1)!}$

$= \frac{4 \times 3!}{3!} \times \frac{7 \times 6 \times 5!}{2! \times 5!} + \frac{4 \times 3 \times 2!}{2! \times 2!} \times \frac{7 \times 6!}{6!} = 4 \times 21 + 6 \times 7 = 84 + 42 = 126$

Example 15.20

A football team of 11 is to be chosen from 15 players. How many different teams can be selected if:

a) there is no condition (no restriction)?

b) the captain was chosen before and must be in the team?

c) the captain was chosen before must be in the team and one of the other players is injured and cannot play?

Solution

a) The number of different teams possible is:

$\binom{15}{11} = \frac{15!}{11! \times (15-11)!} = \frac{15 \times 14 \times 13 \times 12 \times 11!}{11! \times 4!} = \frac{15 \times 14 \times 13 \times 12}{4!}$

$$= \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365$$

b) Since the captain must be in the team, the selection is to have the other 10 players from the 14 available to join the captain. The number of teams possible is

$$\binom{14}{10} = \frac{14!}{10! \times (14-10)!} = \frac{14 \times 13 \times 12 \times 11 \times 10!}{10! \times 4!} = \frac{14 \times 13 \times 12 \times 11}{4!}$$

$$= \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001$$

c) Since the captain must be in the team and one player cannot play because of injury, the selection is to now to choose the other ten players from the 13 available to join the captain.

The number of teams possible is

$$\binom{13}{10} = \frac{13!}{10! \times (13-10)!} = \frac{13 \times 12 \times 11 \times 10!}{10! \times 3!} = \frac{13 \times 12 \times 11}{3!} = \frac{13 \times 12 \times 11}{3 \times 2 \times 1}$$

$= 286$

Application activity 15.3

1. Evaluate: a) C_2^4 b) C_3^9 c) C_4^{16}
2. How many committees of three students can be selected from 20 students
3. In how many ways can a team of 2 men and 3 women be selected in a group of 6 men and 7 women?
4. A basketball team of 6 is to be chosen from 11 available players. In how many ways can this be done if
   a) there are no restrictions?
   b) 3 of the players are automatic selections?
   c) 3 of the players are automatic selections and 2 other players are injured and cannot play?
5. In how many ways can 9 people be placed in 3 cars which can take 2, 3 and 4 passengers respectively, assuming that the seating arrangement inside the cars is not important
6. In how many ways can 12 people be divided into
   a) two groups each with 6 people?
   b) three groups each with 4 people?
7. Find the number of ways in which 6 coins can be distributed between 3 people if the coins are
   a) all different b) indistinguishable.

8. If 5 straight lines are drawn in a plane with no two parallel and no three concurrent, how many different triangles can be formed by joining sets of 3 of the points of intersection?

9. Twelve points in a plane are such that 4 of them lie on a straight line but no other set of 3 or more points are collinear. What is the maximum number of different straight lines which can be drawn through pairs of the given points?

10. In how many ways can 5 identical rings be placed on the 4 fingers of one hand?

11. a) How many distinct non-negative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 10$ exist?

 b) How many positive integer solutions to the equation in part (a) exist?

Pascal's triangles

It is sometimes necessary to expand expressions such as $(x + y)^4$ but the multiplication is tedious when the power is three or more. We now describe a far quicker way of obtaining such expansions.

Consider the following expansions:

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

The first thing to notice is that the powers of x and y in the terms of each expansion form a pattern. Looking at the expansion of $(x + y)^4$, we see that the first term is x^4 and then the power of x decreases by 1 in each succeeding term while the power of y increases by 1. For all the terms the **sum of the powers** of x and y is 4 and the expansion ends with y^4. Notice also that the number of terms is 4 + 1 = 5.

In general, for the expansion of $(x + y)^n$ where n = 1, 2, 3, ...

As we move from left to right across the expansion, the powers of x decrease by 1 while the powers of y increase by 1.

The sum of the power of x and y in each term is n.

The number of terms in the expansion is n + 1.

To construct the Pascal's triangle, start with 1 at the top then continue placing numbers below it in a triangular pattern. Each number is the two numbers above it added together except for the edges, which are all ones.

The following is the table showing Pascal'triangle. It is helpful in finding the coefficients of the terms.

Power	Coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
	...

This array is called Pascal's triangle and clearly it has a pattern.

Pascal's triangle is a triangular arrangement of numbers with a 1 at the top and at the beginning and end of each row, with each of the other numbers being the sum of the two numbers above it.

Pascal's triangle can be written down to as many rows as needed. Using Pascal's triangle to expand $(x + y)^n$, we go as far as row $(n+1)^{th}$ row.

Example 15.21

Expand $(x + y)^5$.

Solution

$$(x + y)^5 = 1x^5y^0 + 5x^4x^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$$
$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Example 15.22

Expand $(2x - 3)^4$.

Solution

Use Pascal's triangle:

$(x + y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Replacing x by $2x$ and y by -3 gives

$$(2x - 3)^4 = (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$$
$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Application activity 15.4

Expand:

1. $(x + 3)^3$
2. $(x - 2)^4$
3. $(x + 1)^4$
4. $(2x + 1)^3$
5. $(x - 3)^5$
6. $(p - q)^4$
7. $(2x + 3)^3$
8. $(x - 4)^5$
9. $(3x - 1)^4$
10. $(1 + 5a)^4$

Binomial expansion

Expanding expressions such as $(1 + x)^{20}$, the coefficients of the terms can be obtained from the Pascal's triangle, but the construction of the triangular array would be tedious. However the **combination notation** could give a more general method to expand powers of any binomial bracket without constructing that triangular array. Thus, if $(x + y)^6$ is to be expanded, we have:

$$(x + y)^6 = \binom{6}{0} x^6y^0 + \binom{6}{1} x^5y^1 + \binom{6}{2} x^4y^2 + \binom{6}{3} x^3y^3 + \binom{6}{4} x^2y^2 + \binom{6}{5} x^1y^5 + \binom{6}{6} x^0y^6$$

$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

This argument can be generalized as follows: if *n* is any positive integer, $(x + y)^n$ can be expanded to give a series of terms in descending power of *x* and ascending power of *y* where the term containing x^{n-n} has the coefficient $\binom{n}{r}$. Thus we have:

$$(x + y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1} x^{n-1}y^1 + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{r} x^{n-r}y^r + \ldots + \binom{n}{n}x^{n-n}y^n$$

In sigma (summation) notation, this gives us the theorem:

For every integer $n \geq 1$:

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}\, y^r$$

Example 15.23

Write down the first three terms of the expansion in ascending powers of x:

(a) $\left(1 - \frac{x}{2}\right)^{10}$ (b) $(3 - 2x)^8$

Solution

(a) Using the binomial expansion and replacing x by 1; y by $-\frac{x}{2}$ and n by 10 we get:

$$\left(1 - \frac{x}{2}\right)^{10} = \binom{10}{0} (1)^{10}\left(-\frac{x}{2}\right)^0 + \binom{10}{1} (1)^9 \left(-\frac{x}{2}\right)^1 + \binom{10}{2} (1)^8 \left(-\frac{x}{2}\right)^2 + \ldots$$

$$= 1 + 10\left(-\frac{x}{2}\right) + \frac{10 \times 9}{2!} \left(\frac{x^2}{4}\right) + \ldots = 1 - 5x + \frac{45}{4}x^2 + \ldots$$

(b) Using the result for $(x + y)^n$ and replacing x by 3; y by $-2x$ and n by 8 gives: $(3 - 2x)^8 = \sum_{r=0}^{8} \binom{8}{r} (3)^{8-r}.(-2x)^r$. Therefore, the first three terms of this series $(r = 0,1,2)$ are

$$\binom{8}{r}3^8 (-2x)^0 + \binom{8}{1}3^7 (-2x)^1 + \binom{8}{2}3^6(-2x)^2 = 3^8 + 8(3^7)(-2x) + 28(3^6)4x^2$$

$$= 6561 - 34992x + 81648x^2$$

Example 15.24

Write down the expansions of each of the following:

(a) $(a + b)^4$ (b) $(a - b)^3$ (c) $(x + 2y)^5$

Solution

(a) $(a + b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$

$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

(b) $(a - b)^3 = \binom{3}{0}a^3b^0 - \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 - \binom{3}{3}a^0b^3 = a^3 - 3a^2b + 3ab^2 - b^3$

(c) $(x + 2y)^5$

$= \binom{5}{0}x^5(2y)^0 + \binom{5}{1}x^4(2y)^1 + \binom{5}{2}x^3(2y)^2 + \binom{5}{3}x^2(2y)^3 + \binom{5}{4}x^1(2y)^4 + \binom{5}{5}x^0(2y)^5$

$= x^5 + 5x^4(2y) + 10x^3(4y^2) + 10x^2(8y^3) + 5x(16y^4) + 32y^5$

$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

Application activity 15.5

1. Use the binomial theorem to expand each of the following:
   a) $(x + y)^4$; b) $(a - b)^7$
   c) $(2 + p^2)^6$; d) $(2h - k)^5$;
   e) $\left(x + \frac{1}{x}\right)^3$; f) $\left(z - \frac{1}{2z}\right)^8$
2. Expand and simplify $(2x + \frac{1}{x^2})^5 + (2x - \frac{1}{x^2})^5$
3. Find the value of n if the coefficient of x^3 in the expansion of $(2 + 3x)^n$ is twice the coefficient of x^2.
4. The coefficient of x^5 in the expansion of $(1 + 5x)^8$ is equal to the coefficient of x^4 in the expansion of $(a + 5x)^7$. Find the value of a.
5. Use the expansion of $(2 - x)^5$ to evaluate 1.98^5 correct to 5 decimal places.
6. In the expansion of each of the following, find the coefficient of the specified power of x:
   a) $(1 + 2x)^7$, x^4 b) $(5x - 3)^7$, x^4
   c) $(3x - 2x^3)^5$, x^{11} d) $\left(x - \frac{2}{x}\right)^8$, x^2
   e) $\left(x^2 + \frac{4}{x}\right)^{10}$, x^{-1}

Summary

1. **Combinatorics** (combinatorial analysis) is the area of mathematics concerned with counting strategies to calculate the ways in which objects can be arranged to satisfy given conditions.
2. **Venn diagrams** are a way of representing sets in a closed curve. They are usually circular or oval shaped.
3. The **complement** of a set, say A is the set written A′ or $\overline{A}$ and is an element in the universal set but not in the set.
4. **Intersection** of sets A and B is denoted by $A \cap B$. This represents the set of all elements common to both A and B.
5. **Union** of set A and B is denoted by $A \cap B$. This represents the set of all elements belonging to A or B or both A and B.
6. **Disjoint sets** are sets with no elements in common. If A and B are disjoint set, then $A \cap B = \varnothing$. Also A and B are said to be mutually exclusive.
7. A **contingency table** is a method of presenting the frequencies of the outcomes from an experiment in which the observations in the sample are categorised according to two criteria.
8. The **factorial** notation of n integers denoted by is the product of the first consecutive integers and is read as 'factorial n'.
9. The number of arrangements of r objects, taken from n unlike objects, can be considered as the number of ways of filling r places in order by the n given objects is given by

   $$p_r^n = \frac{n!}{(n-r)!}$$
10. In general, the number of different ways of arranging n objects on a circle is $(n-1)!$
11. In general, the number of ways of arranging in a line n objects of which p of one type are alike, q of a second type are alike, r of a third type are alike, and so on, is $\frac{n!}{p!q!r!..}$.
12. In general, the number of combinations of r objects from n unlike objects is

    $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Condition $n \geq r$
13. Binomial expansion: if n is any positive integer, can be expanded to give a series of terms in descending power of x and ascending power of y where the term containing x^{n-r} has the coefficient .

Topic area: Statistics and probability

Sub-topic area: Combinatorial analysis and probability

Unit 16

Elementary probability

Key unit competence

Use counting techniques and concepts of probability to determine the probability of possible outcomes of events occurring under equally likely assumptions.

16.0 Introductory activity

When a card is selected from an ordinary deck of 52 cards, one assumes that the deck has been shuffled, and each card has the same chance of being selected. Let A be the event of selecting a black card,

a) If n is the number of black cards in the pack, what is the value of n?

b) Calculate the value $P(A)=\dfrac{n}{number\ of\ all\ cards}$

c) If $P(A)$ is the probability of selecting a black card, deduce the definition of probability for any event A.

16.1 Concepts of probability

Activity 16.1

In groups, discuss the following.

Have you ever watched people gambling? What do they rely on to win? What are the problems associated with gambling?

Present your findings to the rest of the class.

Many events cannot be predicted with total certainty. The best we can say is how **likely** they are to happen, using the idea of probability.

Tossing a coin

When a coin is tossed, there are two possible outcomes:

heads (H) or

tails (T)

We say that the probability of the coin landing **H** is $\frac{1}{2}$.

And the probability of the coin landing **T** is $\frac{1}{2}$.

Throwing dice

When a single die (plural dice) is thrown, there are six possible outcomes: **1, 2, 3, 4, 5, 6**.

The probability of obtaining any one of them is $\frac{1}{6}$.

Probability

In general:

$$\text{Probability of an event happening} = \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$$

Random experiment

Rolling (tossing) a die is an example of a **random experiment** and probability is the study of such random experiments. When we roll a die, we know that the set of possible outcomes is S = {1, 2, 3, 4, 5, 6}, called the **sample space**. We have no idea exactly which of the elements of S will appear in any toss but we know intuitively that each of these outcomes is equally likely. That is, a '2' is no more likely to appear than a '1', which is non more likely to appear than a '3', and so on. If this experiment is performed on n separate occasions and 't' is the number of times a '2' appears, we know from observation that the ratio $\frac{t}{n}$ becomes close to $\frac{1}{6}$ as n increases. We can define probability theory as the study of the chance of an event happening.

Example 16.1

What is the chance of getting a 4 when a die is rolled?

Solution

Number of ways it can happen: 1 (there is only 1 face with a 4 on it)
Total number of outcomes: 6 (there are 6 faces altogether)

So the probability = $\frac{1}{6}$

Example 16.2

There are 5 marbles in a bag: 4 are blue, and 1 is red. What is the probability that a blue marble gets picked?

Solution

Number of ways it can happen: 4 (there are 4 blues)

Total number of outcomes: 5 (there are 5 marbles in total)

So the probability = $\frac{4}{5}$ = 0.8

Sample space and events

The set S of all possible outcomes of a given experiment is called the **sample space**. A particular outcome, i.e., an element of S, is called a sample point. Any subset of the sample space is called an **event**. The event {a} consisting of a single element of S is called a simple event.

Experiment or trial: an action where the result is uncertain.

Tossing a coin, throwing dice, seeing what fruits people prefer are all examples of experiments.

Sample space: all the possible outcomes of an experiment

Choosing a card from a deck

There are 52 cards in a deck (not including Jokers)

So the sample space is all 52 possible cards: {Ace of Hearts, 2 of Hearts, and so on... }

Event: a single result of an experiment

- Getting a Tail when tossing a coin is an event
- Rolling a "5" is an event.

An event can include one or more possible outcomes:

- Choosing a "King" from a deck of cards (any of the 4 Kings) **is** an event
- Rolling an "even number" (2, 4 or 6) is also an event

Anne wants to see how many times a "double" comes up when throwing 2 dice.

Each time Anne throws the 2 dice is an **experiment**. It is an experiment because the result is uncertain.

The event Anne is looking for is a "double", where both dice have the same number. It is made up of these 6 sample points:

(1,1), (2,2), (3,3), (4,4), (5,5) and (6,6) or S ={(1,1), (2,2), ...}

The sample space is all possible outcomes (36 sample points):

(1,1), (1,2), (1,3), (1,4) ... (6,3), (6,4), (6,5), (6,6)

Probability of an event under equal assumptions

The probability of an event E, denoted by P(E) or Pr(E), is a measure of the possibility of the event occurring as the result of an experiment.

If the sample space S is finite and the possible outcomes are all equally likely, then the probability of the event E is equal to $\frac{|E|}{|S|} = \frac{n(E)}{n(S)}$ where |E| and |S| denote the number of elements in E and S respectively.

$$P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{|E|}{|S|}$$

The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.

Theorem

Suppose that an experiment has only a finite number of equally likely outcomes. If E is an event, then $0 \leq P(E) \leq 1$.

Proof: Since E is a subset of S, the set of equally likely outcomes,

then $0 \leq |E| \leq |S|$. Hence $0 < \frac{|E|}{|S|} \leq 1$ or $0 \leq P(E) \leq 1$

Note: if E = S, then clearly |E| = |S| and P(E) = P(S) = 1 (the event is certain to occur), and if E = ∅, then |E| = 0 and P(E) = P(∅) = 0 (the event cannot occur).

Example 16.3

A coin is tossed two times. Find the probability of obtaining

(a) A = {two heads} (b) B = {one head and one tail}

Solution

The sample space is S = {HH, HT, TH, TT},

A = {HH}, B = {HT, TH},

(a) $P(A) = \frac{|A|}{|S|} = \frac{1}{4}$ (b) $P(B) = \frac{|B|}{|S|} = \frac{2}{4} = \frac{1}{2}$

Example 16.4

A letter is chosen from the letters of the word "MATHEMATICS". What is the probability that the letter chosen is an "A"?

Solution

Since two of the eleven letters are "A", the probability of choosing a letter "A" is $\frac{2}{11}$.

Complementary event

If E is an event, then E′ is the event which occurs when E does not occur. Event E and E′ are said to be **complementary events**.

Theorem

$P(E') = 1 - P(E)$ or $P(E) = 1 - P(E')$

Proof:

$$P(E') = \frac{|E'|}{|S|} = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|}$$

Thus $P(E') = 1 - P(E')$ and clearly $P(E)=1- P(E')$. Also $P(E) + P(E') = 1$.

Example 16.5

An ordinary die of 6 sides is rolled once. Determine the probability of:

a) obtaining 5

b) not obtaining 5

c) obtaining 3 or 4

d) not obtaining 3 or 4

Solution

The sample space of possible outcomes is S = {1, 2, 3, 4, 5, 6}

a) $P(5) = \frac{1}{6}$

b) $P(\text{not a } 5) = 1 - \frac{1}{6} = \frac{5}{6}$

c) $P(3 \text{ or } 4) = \frac{2}{6} = \frac{1}{3}$

d) $P(\text{not } 3 \text{ or } 4) = 1 - \frac{1}{3} = \frac{2}{3}$

Consider two different events, A and B. which may occur when an experiment is performed. The event $A \cup B$ is the event which occurs if A or B or both A and B occurs, i.e.. at least one of A or B occurs.

The event $A \cap B$ is the event which occurs when both A and B occurs.

The event $A - B$ is the event which occurs when A occurs and B does not occur.

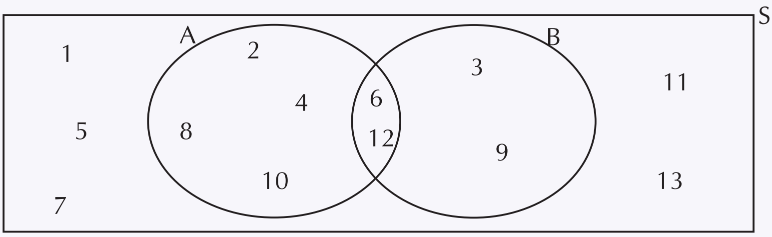
The event A' is the event which occurs when A does not occur.

Example 16.6

An integer is chosen at random from the set $S = \{x | x \in \mathbb{Z}^+, 0 < x < 14\}$. Let A be the event of choosing a multiple of 2 and B the event of choosing a multiple of 3. Find

(a) $P(A \cup B)$; (b) $P(A \cap B)$; (c) $P(A - B)$.

Solution



From the diagram

(a) $P(A \cup B) = \frac{8}{13}$ (b) $P(A \cap B) = \frac{2}{13}$

(c) $P(A - B) = \frac{4}{13}$

Permutations and combinations in probability theory

Activity 16.2

Discuss the following scenario in groups and present your findings to the rest of the class.

If there are 4 women and 3 men and you wanted them seated in a row, what is the probability that the men will be seated together?

Permutations and combinations can be used to find probabilities of various events particularly when large sample sizes occur. In everything we do, we have to use the formula

$$P(E) = \frac{\text{number of elements of event E}}{\text{the total number of possible outcomes}} = \frac{n(E)}{n(S)}$$

Example 16.7

A bag contains 6 blue balls, 5 green balls and 4 red balls. Three balls are selected at random without replacement. Find the probability that

a) they are all blue b) 2 are blue and 1 is green

c) there is one of each colour.

Solution

The number of all possible outcomes is $\binom{15}{3} = \frac{15!}{3! \times 12!} = \frac{15 \times 14 \times 13}{3!} = 455$

a) The number of ways of obtaining 3 blue balls is

$$\binom{6}{3} \times \binom{5}{0} \times \binom{4}{0} = \frac{6!}{3!3!} \times 1 \times 1 = \frac{6 \times 5 \times 4}{3!} = \frac{120}{6} = 20$$

Thus, P (all are blue) $= \frac{20}{455} = \frac{4}{91}$

b) The probability of obtaining 2 blue balls and one green ball is:

$$\binom{6}{2} \times \binom{5}{1} \times \binom{4}{0} = \frac{6!}{2!4!} \times 5 \times 1 = \frac{6x5}{2!} \times 5 = \frac{150}{2} = 75$$

Thus, P(2 blue and 1 green) = $\frac{75}{455} = \frac{15}{91}$

c) The probability of obtaining 1 blue ball, 1 green ball and 1 red ball is:

$$\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} = \frac{6!}{1!5!} \times \frac{5!}{1!4!} \times \frac{4!}{1!3!} = 6 \times 5 \times 4 = 120$$

Thus, P (1 blue, 1 green and 1 red) = $\frac{120}{455} = \frac{24}{91}$

Example 16.8

If 4 people A, B, C, D sit in a row on a bench, what is the probability that A and B sit next to each other?

Solution

The number of ways of arranging 4 people in a row is $|S| = 4! = 24$.

The number of ways of arranging 4 people so that A and B are next to each other is $|X| = 2 \times 3! = 12$.

Thus, P(A and B sit next to each other) is $\frac{|X|}{|S|} = \frac{12}{24} = \frac{1}{2}$.

Example 16.9

If 5 cards are selected at random from an ordinary deck of 52 cards, find the probability that exactly 2 of them are aces.

Solution

The number of ways of selecting 2 aces from the 4 aces is $\binom{4}{2}$.

The number of ways of selecting 3 non-aces from the 48 non-aces is $\binom{48}{3}$.

Therefore the number of ways of selecting 5 cards of which exactly 2 are aces is $|A| = \binom{4}{2} \times \binom{48}{3}$.

The number of ways of selecting 5 cards from 52 is $|S| = \binom{52}{5}$.

Thus the required probability is

$$\frac{|A|}{|S|} = \frac{\binom{4}{2} \times \binom{48}{3}}{\binom{52}{5}} = \frac{\frac{4!}{2! \times (4-2)!} \times \frac{48!}{3! \times (48-3)!}}{\frac{52!}{5! \times (52-5)!}} = \frac{\frac{4 \times 3 \times 2!}{2! \times 2!} \times \frac{48 \times 47 \times 46 \times 45!}{3! \times 45!}}{\frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{5! \times 47!}} =$$

$$\frac{\frac{4 \times 3}{2} \times \frac{48 \times 47 \times 46}{6}}{\frac{52 \times 51 \times 50 \times 49 \times 48}{120}} = \frac{6 \times \frac{48 \times 47 \times 46}{6}}{\frac{52 \times 51 \times 50 \times 49 \times 48}{120}}$$

$$= \frac{48 \times 47 \times 46}{1} \times \frac{120}{52 \times 51 \times 50 \times 49 \times 48} = \frac{12453120}{311875200} \approx 0.0399.$$

16.2 Finite probability spaces

Let $S = \{a_1, a_2, a_3, ..., a_n\}$ be a finite sample space. A finite probability space obtained by assigning to each point $a_r \in S$ a real number p_r called the probability of a_r, satisfying the following:

(a) $p_r \geq 0$ for all integers r $1 \leq r \leq n$

(b) $\sum_{r=1}^{n} P_r = 1$

If A is any event, then the probability P(A) is defined to be the sum of the probabilities of the sample points in A.

Example 16.10

A coin is weighted such that heads is three times as likely to appear as tails. Find P(T) and P(H).

Solution

Let P(T) = p, then P(H) = 3p. However P(T) + P(H) = 1. Therefore 4p = 1 or p = $\frac{1}{4}$. Thus P(T)= $\frac{1}{4}$ and P(H) = $\frac{3}{4}$.

Application activity 16.1

1. An unbiased cubic die is thrown. Find the probability that the number showing is

   a) Even; b) prime; c) less than 4

2. A die is loaded in such a way that $P(1) = P(3) = \frac{1}{12}$, $P(2) = P(6) = \frac{1}{8}$ and $P(4) = \frac{1}{2}$. Find the probability that the number appearing is

   a) odd b) even c) prime d) not 3.

3. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability that:

   a) both cards are spades b) at least one card is a spade.

4. The letters of the word FACETIOUS are arranged in a row. Find the probability that:

   a) the first 2 letters are consonants b) all the vowels are together.

16.3 Sum and product laws

Theorem: If A and B are events from a sample space S, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: $P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cup B|}{|S|} =$
$P(A) + P(B) - P(A \cap B)$.

This is known as the addition law of probability.

Mutually exclusive events

Events A and B are said to be mutually exclusive if the events A and B are **disjoint** i.e. A and B cannot occur at the same time.

For mutually exclusive events, $A \cap B = \varnothing$. $P(A \cap B) = P(\varnothing) = 0$; and so the addition law reduces to $P(A \cup B) = P(A) + P(B)$.

Example 16.11

A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find P(A), P(B), $P(A \cap B)$ and $P(A \cup B)$.

Solution

$P(A) = P(\text{an ace}) = \frac{4}{52} = \frac{1}{13}$

$P(B) = P(\text{a spade}) = \frac{13}{52} = \frac{1}{4}$

$P(A \cap B) = P(\text{the ace of spades}) = \frac{1}{52}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Example 16.12

A marble is drawn from an urn containing 10 marbles of which 5 are red and 3 are blue. Let A be the event: the marble is red; and let B be the event: the marble is blue. Find P(A), P(B) and $P(A \cup B)$.

Solution

$P(A) = \frac{5}{10} = \frac{1}{2}$, $P(B) = \frac{3}{10}$ and since the marble cannot be both red and blue,

A and B are mutually exclusive so $P(A \cup B) = P(A) + P(B) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$

Independent events

Two events are independent if the occurrence or non occurrence of one of them does not affect the occurrence of the other. Otherwise, A and B are dependent. For independent events, the probability that they both occur is given by the following product law:

$P(A \cap B) = P(A) \times P(B)$

Example 16.13

Given that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = x$. Find x if A and B are

(a) independent

(b) mutually exclusive.

Solution

(a) If the events are independent, then $P(A \cap B) = P(A) \times P(B) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{2}{5} - \frac{2}{15} = \frac{5 + 6 - 2}{15} = \frac{9}{15} = \frac{3}{5}$

Thus, $x = \frac{3}{5}$

(b) If the events are mutually exclusive, then $A \cap B = \varnothing$ which gives $P(A \cap B) = 0$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$x = \frac{1}{3} + \frac{2}{5} - 0 = \frac{5 + 6}{15} = \frac{11}{15}$

Thus, $x = \frac{11}{15}$

Example 16.14

A factory runs two machines, A and B. Machine A operates for 80% of the time while machine B operates for 60% of the time and at least one machine operates for 92% of the time. Do these machines operate independently?

Solution

The data does not give any clues. However we are given $P(A) = 0.8$, $P(B) = 0.6$ and $P(A \cup B) = 0.92$.

Now $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 + 0.6 - 0.92 = 0.48$ and we have $P(A) \times P(B) = 0.8 \times 0.6 = 0.48$.

Since $P(A \cap B) = 0.48 = P(A) \times P(B)$, then, these machines do operate independently.

Example 16.15

Find the probability that in 3 throws of a fair die, the 3 numbers are all even.

Solution

The probability that the first number is even is $\frac{1}{2}$. The probability that the second number is even is $\frac{1}{2}$. The probability that the third number is even is $\frac{1}{2}$. Since these events are independent, the probability that the 3 numbers are all even is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

This rule is the simplest form of the multiplication law of probability. The extension to events which are not independent will be considered in the next section.

Application activity 16.2

1. If A and B are any two events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{12}$, and $P(A \cap B) = \frac{1}{4}$. Find $P(A \cup B)$.
2. Events A and B are such $P(A - B) = 0.3$, $P(B - A) = 0.4$ and $P(A' \cap B') = 0.1$. Find

   a) $P(A \cap B)$ b) $P(A)$ c) $P(B)$
3. If A and B are independent events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{6}$, find

   a) $P(A \cap B)$; b) $P(A \cup B)$; c) $P(A \cap B')$;

   d) $P(A' \cup B')$
4. A die is biased so that the probability of throwing a six is $\frac{1}{3}$. If the die is thrown twice, find the probability of obtaining

   a) two sixes b) at least one six.
5. An unbiased die is thrown three times. Find the probability of obtaining

   a) 3 sixes; b) exactly 2 sixes; c) at least one six

Summary

1. The set S of all possible outcomes of a given experiment is called the **sample space**.
2. Any subset of the sample space is called an **event**. The event {a} consisting of a single element of S is called a simple event.
3. The probability of an event E, denoted by P(E) or Pr(E), is a measure of the possibility of the event occurring as the result of an experiment.
4. The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.
5. If E is an event, then E′ is the event which occurs when E does not occur. Event E and E′ are said to be **complementary events**.
6. A finite probability space obtained by assigning to each point $a_r \in S$ a real number p_r called the probability a_r, satisfying the following:
   a. $p_r \geq 0$ for all integers r, $1 \leq r \leq n$
   b. $\sum_{r=1}^{n} P_r = 1$
7. Events A and B are said to be **mutually exclusive** if the events A and B are **disjoint** i.e. A and B cannot occur at the same time.
8. Two events are **independent** if the occurrence or non occurrence of one of them does not affect the occurrence of the other.

Practice Tasks

Practice Task 1

1. Use the truth table to show that $\sim (p \vee q)$ is logically equivalent to $(\sim p) \wedge (\sim q)$.
2. Negate each of the following statements
   a) $\forall\ x \in \mathbb{R}: x^2 > 0$
   b) $\forall\ x \in \mathbb{R}; \exists n \in \mathbb{N}$ such that $x < n$
3. Find the greatest value of each expression and the value of θ between 0 and 360
   a) $\sin\theta \cos 15° - \cos\theta \sin 15°$
   b) $\sin\theta \sin 30° + \cos\theta \cos 30°$
   c) $\cos\theta \cos 60° - \sin\theta \sin 60°$
   d) $\sin 60° \cos\theta - \cos 60° \sin\theta$
4. Solve the following equations
   a) $8^{1-x} = 4^{2x+3}$
   b) $a^{\frac{3}{2}} = 8$
   c) $a^{\frac{2}{3}} = 16$
5. Find the values of p for which the given equation has real roots
   a) $x^2 + (p + 3)x - 4p = 0$
   b) $x^2 + (3 - p)x + 1 = 0$
6. Evaluate the following limits
   a) $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$
   b) $\lim\limits_{x \to 2} \dfrac{2 - \sqrt{x + 2}}{x - 2}$
   c) $\lim\limits_{x \to 4} \dfrac{\sqrt{2x + 1} - 3}{x - 4}$
   d) $\lim\limits_{x \to 4} \dfrac{\sqrt[3]{x + 2}}{\sqrt[3]{x + 3}}$
   e) $\lim\limits_{x \to +\infty} \sqrt{4x^2 - x + 7} - 2x$

7. Find any asymptotes of the function f if $f(x) = \frac{x+2}{x^2-4}$

8. Write in interval set form the solution set of the inequality $|-3x + \frac{1}{2}| \le \frac{1}{4}$

9. In $\mathbb{R} \times \mathbb{R}$, solve $\begin{cases} 4x + y = 5 \\ \log x + \log y = 0 \end{cases}$

10. A is an acute angle and $\sin A = \frac{7}{25}$, B is obtuse and $B = \frac{4}{5}$. Find an exact expression for

    a) $\sin(A + B)$

    b) $\cos(A + B)$

    c) $\tan(A + B)$

11. Find the derivative of the following function using the definition of the derivative: $g(t) = \sqrt{t}$

12. Find the set of values of a for which $x^2 + 3ax + a$ is positive for all real values of x.

13. How many different ways can a committee of 10 people composed of 7 women be chosen from a list made up of 8 women and 12 men?

14. Find a second degree polynomial P(x) such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$ where P' and P'' are first and second derivatives of P respectively.

15. How many distinct permutations can be made from the letters of the word "**INFINITY**"?

16. Given the function $f(x) = \frac{x+1}{(x-3)^2}$. Determine:

    a) the domain of f

    b) the x-intercept(s) and y-intercept(s)

    c) all asymptotes to the curve of the function f.

    d) the first derivative $f'(x)$ and the second derivative $f''(x)$

    e) the extrema point(s) (local minimum or local maximum)

    f) interval(s) on which f is increasing or decreasing.

    g) inflection point(s)

    h) interval(s) of upward or downward concavity.

17. Find the range of each of the following functions.

    a) $f(x) = 2x - 3$ for $x \ge 0$
    b) $f(x) = x^2 - 5$ for $x \le 0$
    c) $f(x) = 1 - x$ for $x \le 1$
    d) $f(x) = \frac{1}{x}$ for $x \ge 2$

18. The probability that a patient recovers from a delicate heart operation is 0.8. What is the probability that

    a) exactly 2 of the next 3 patients who have this operation survive

    b) all of the next 3 patients who have this operation survive?

19. The number of incorrect answers on a true–false competency test for a random sample of 15 students were recorded as follows:

2, 1, 3, 0, 1, 3, 6, 0, 3, 3, 5, 2, 1, 4, and 2.

Determine the:

a) mean

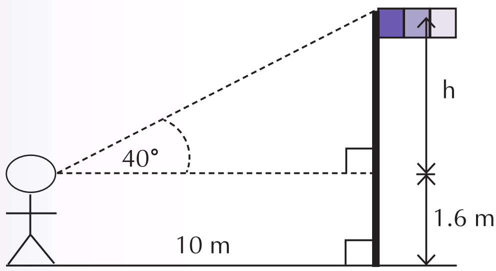
b) median

c) mode

d) sample standard deviation

20. a) Determine $\tan 2\theta$ if $\tan \theta = \frac{3}{4}$ and θ is the acute angle

b) A person stands 10 m away from a flagpole and measures an angle of elevation of 40° from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are at a vertical distance of 1.6 m from the ground. What is the height of the flagpole?



Practice Task 2

1. Determine the centre and radius for the following equations of circles:

   a) $x^2 + y^2 + 8x - 2y - 8 = 0$ b) $x^2 + y^2 + x + 3y - 2 = 0$

   c) $x^2 + y^2 + 6x - 5 = 0$

2. If 3 books are picked at random from a shelf containing 5 novels, 3 poetry books, and one dictionary, what is the probability that

   a) the dictionary is selected?

   b) 2 novels and 1 poetry book are selected?

3. A square matrix A is such that $A^2 = I - 5A$. Express A^{-1} in linear form $\alpha A + \beta I$, where α and β are scalars

4. Solve the following inequalities

   a) $x < 2x + 4 < 2 - x$ b) $-5 < x^2 - 9 < x - 3$

   c) $\frac{x - 1}{2 + x} > 1$ d) $(3x - 2)(2x + 1) < 6x - 3$

5. Let * be a binary operation defined on the set $\mathbb{Z}$ of all integers by $x * y = x + y + 3$. Determine whether the operation is commutative, and whether there is an identity element.

   Can you find a symmetric (inverse) of any integer?

6. Find the exact value of each expression, leaving your answer in surd form where necessary

   a) $\cos 35° \cos 65° - \sin 35° \sin 65°$

   b) $\sin 40° \cos 10° - \cos 40° \sin 10°$

   c) $\cos 75°$

   d) $\tan 105°$

   e) $\sin 165°$

   f) $\cos 15°$

7. Simplify each of the following expressions

   a) $\sin \theta \cos 2\theta + \cos \theta \sin 2\theta$

   b) $\cos \alpha \cos (90° - \alpha) - \sin \alpha \sin (90° - \alpha)$

8. If $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$ find

   a) A^{-1} b) B^{-1} c) $(AB)^{-1}$ d) $(BA)^{-1}$

   e) $A^{-1}B^{-1}$ f) $B^{-1}A^{-1}$

9. Solve

   a) $2^{3x} = 3^{2x - 1}$ b) $|x + 2| > |2x - 1|$

10. The amount A(t) gram of radioactive material in a sample for t years is given

by $A(t) = 80\left(2^{-\frac{t}{100}}\right)$.

a) Find the amount of material in the original sample.

b) Calculate the half-life of the material. (Half-life: the time taken for half of the original material to decay.)

c) Calculate the time taken for the material to decay to 1.

11. Find the coordinates of the vertex of the graph of $y = 2x^2 + 3x - 5$ and the equation of its axis of symmetry.

12. Find the values of k for which the equation $x^2 + (k + 1)x + 1 = 0$ has:

a) two distinct real roots b) no real roots

13. If $f(x) = x + 2$ and $g(x) = 2x + 3$, Find

(a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$ (c) $(f \circ f)(x)$

(d) $(g \circ g)(x)$

14. State whether the following three vectors $\vec{v}_1 = \binom{2}{7}$, $\vec{v}_2 = \binom{1}{8}$, $\vec{v}_3 = \binom{4}{5}$ are linearly dependent or linearly independent.

15. Consider the set of vectors from $\mathbb{R}^2$, $S = \{\vec{v}, \vec{w}\}$, where $\vec{v} = \binom{3}{4}$ and $\vec{w} = \binom{2}{5}$.

a) Show that S spans $\mathbb{R}^2$.

b) Show that S is linearly independent.

16. A card is drawn at random from an ordinary pack of 52 playing cards.

(a) Find the probability that the card drawn is

(i) the four of spades

(ii) the four of spades or any diamond

(iii) not a picture card (Jack or Queen or King) of any suit.

(b) The card drawn is the three of diamonds. It is placed on the table and a second is drawn. What is the probability that the second card drawn is not a diamond?

17. Find the perimeter of an isosceles triangle whose base angle is 70° and whose base side is 40 cm.

18. Given $f(x) = x^2$ and $g(x) = 2 - x$. Find

a) $(f \circ g)(x)$ and find its domain and range

b) $(g \circ f)(x)$ and find its domain and range

19. Given that $f(x) = x^3 + ax^2 + bx + 3$ and $\frac{d}{dx}f(x)$, each has $x - 1$ as a factor

a) find the values of a and b b) factorize f(x)

20. In how many ways can a party of seven persons arrange themselves

a) in a row of 7 chairs b) in a circular round of 7 chairs

Practice Task 3

1. Determine the shortest distance between a point (1, 2) and a line of equation $3x - 4y + 8 = 0$.
2. In a class of 20 students, 4 of the 9 boys and 3 of the 11 girls are in the athletics team. A student from the class is chosen to be in the 'Kigali tour' race on Sports Day. Find the probability that the student chosen is
   a) in the athletics team
   b) a female
   c) a female member of the athletics teams
   d) a female or in the athletics team
3. Find the equation of the tangents to the curve $y = x^2 - 6x + 5$ at each of the points where the curve crosses the x-axis. Find also the coordinates of the point where these tangents meet.
4. How many arrangements of the letters in the word **MIYOVE** start with a consonant?
5. Given a circle of equation $x^2 + y^2 + 4x + 6y - 3 = 0$ and a line of equation $L \equiv 3x - 4y + 14 = 0$:
   a) determine the centre and radius of the circle
   b) determine the shortest distance between the centre of the circle and the line L
   c) state whether or not the line is the tangent to the circle
   d) if the line L is the tangent to the circle, determine the point of tangency.
6. How many odd numbers between 2000 and 3000 can be formed from the digits 1, 2, 3, 4, 5 and 6 using each of them only once in each number?
7. Evaluate the following limits
   a) $\lim_{x \to 3} \frac{x^2 - 4x + 4}{x - 2}$ b) $\lim_{x \to 2} \frac{3x^2 - 5x - 2}{x^2 - 4}$
   c) $\lim_{x \to +\infty} \frac{2x^2 + 7x - 3}{x^2 - 2x + 1}$ d) $\lim_{x \to +\infty} (x - \sqrt{x^2 + 4x})$
   e) $\lim_{x \to -\infty} \frac{\sqrt{9x^2 + 6x + 1}}{6x + 1}$
8. In how many of the possible permutations the letters of the word **MATTER** are the two Ts
   a) together? b) separated?
9. Write down the first three terms in the expansion in ascending powers of x of
   a) $(3 - 2x)^8$ b) $(1 - \frac{x}{2})^{10}$

10. Consider the curve given by $y = 3x^4 - 6x^2 + 2$. Find

  a) the slope of the curve at the point whose x-coordinate is 2

  b) equation for the tangent at the point $(1, -1)$

  c) the point(s) on the curve at which the tangent is horizontal.

11. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function given by $f(x) = x^4 - 4x^3 + 5$. Find where the graph of the function is increasing or decreasing.

12. Given the data 2, 4, 5, 8, 11. Determine the:

  a) mean b) median

  c) mode d) range

  e) variance f) standard deviation

13. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^4 - 4x^3 + 5$.

  a) Find the inflection number(s) and inflection point(s)

  b) Find where the graph of f is curving up or curving down.

14. Given the function $f(x) = \begin{cases} \dfrac{|x-3|}{x^2-9} & \text{if } x < 3 \\ k & \text{if } x \geq 3 \end{cases}$.

  Find all values of k that makes the function f(x)continuous at $x = 3$.

15. Find the points on the graph of $f(x) = \frac{1}{3}x^3 + x^2 - x - 1$ where the slope is

  a) 0 b) 2 c) -1

16. Manikuzwe observes a top of a tree at an angle of elevation 20° . When he advances 75 m towards the base of the tree, the angle of elevation changes to 40°. Find the height of the tree.

17. Find the largest possible value for xy given that x and y are both non-negative numbers and $x + y = 20$.

18. A tower of height 100 m casts a shadow of 120 m. Find the measure of the angle of elevation of the sun.

19. a) In a single throw of two dice, determine the probability of getting a total sum of 2 or 4.

  b) The letters of the word "**KWIZERA**" are arranged at random. Find the probability that the vowels may occupy the even places.

20. Given the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$:

  a) find its inverse A^{-1}

  b) use it to solve the system $\begin{cases} x + 2y = 5 \\ 3x + 4y = 11 \end{cases}$

References

1. Boutriau-Philipe, E. et al. (1994) **Savoir Faire en Mathйmatique**, 2e Annйe, 8e edition, De Boeck-Wesmael, Bruxelles,1994
2. Delord, R., (1992) **Mathйmatiques, 4e annйe**,. Hachette,1992
3. Rich B & Thomas C. (2009). **Schaum's Outline, Geometry**
4. Rwanda Education Board (2015). **Advanced Level Mathematics Syllabus for Science Combinations**